# Exercise sheet 3

meeting on **21/04/2020**

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**Exercise 13** [Lemma 3.1.3] Let $X, Y \subseteq \mathbb{A}^n(K), S \subseteq K[x_1, \ldots, x_n]$ and $a_1, \ldots, a_n \in K$. Then show the following:

a) If $X \subseteq Y$, then $I(X) \supseteq I(Y)$.

b) $I(\emptyset) = K[x_1, \ldots, x_n]$ and $I(\mathbb{A}^n(K)) = \{0\}$ for infinite $K$.

c) $I(\{(a_1, \ldots, a_n)\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

d) $I(V(S)) \supseteq S$ and $V(I(X)) \subseteq X$.

e) $I(X)$ is radical.

**Exercise 14** Consider the system of equations

$$x^2 + 2y^2 = 3,\ x^2 + xy + y^2 = 3.$$

a) If $I$ is the ideal generated by these equations, find generators for $I \cap K[x]$ and $I \cap K[y]$.

b) Compute $V(I)$.

c) Which elements in $V(I)$ are rational? What is the smallest field $K$ such that all zeros are contained in $K^2$?

d) Repeat the previous tasks with the equations $x^2 + 2y^2 = 2,\ x^2 + xy + y^2 = 2$.

**Exercise 15** Given are the parametric surfaces $S_1$ and $S_2$ defined by

$S_1 :\ x = u\,v,\ y = u^2,\ z = v^2;$

$S_2 :\ x = u\,v,\ y = u\,v^2,\ z = u^2.$

a) Visualize the surfaces.

b) Find the defining equations in $\mathbb{C}[x, y, z]$ of the algebraic closure $V_i$ of $S_i$.

c) Are there any points in $V_i \setminus S_i$? In the affirmative, compute all of them.

**Exercise 16** Let us consider space curves parametrized by $x_i = f_i(t)$, where $f_1, \ldots, f_n \in \mathbb{C}[t]$, and its defining ideal

$$I = \langle x_1 - f_1(t), \ldots, x_n - f_n(t) \rangle \subseteq \mathbb{C}[t, x_1, \ldots, x_n].$$

a) Prove that the parametric equations fill up the full variety $V(I_1) \subset \mathbb{C}^n$.

b) Show that this is in general not true if

  i) the defining polynomials $f_i$ are replaced by rational functions, i.e. $f_1, \ldots, f_n \in \mathbb{C}(t)$;

  ii) the algebraic set $V(I_1)$ is considered in the reals.

**Exercise 17** a) Show that for $f, g \in K[x, y]$ it follows that

$$Res_x(f, g) \in I_1 = \langle f, g \rangle \cap K[y].$$

b) What do you obtain for the generators of $I_1$ in the cases of

$$f_1 = x\,y - 1,\ g_1 = x^2 + y^2 - 4$$

and

$$f_2 = f_1,\ g_2 = y\,x^2 + y^2 - 4?$$

Draw some connection to the Extension Theorem (Theorem 4.1.1 in the lecture notes).

**Exercise 18** Find the Zariski closure of the following sets in $\mathbb{A}^2(\mathbb{C})$:

a) $\{(n^2, n^3) \mid n \in \mathbb{N}\}$;

b) the natural numbers $\mathbb{N}$;

c) the projection $\pi_1(V(x\,y - 1))$.

In the last case verify Theorem 4.1.4 from the lecture notes and find such a $W$.