Exercise sheet 2

meeting on **24/03/2020**

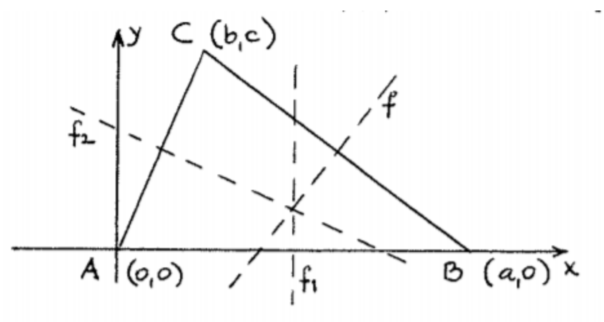
Exercise 7 Let us consider algebraic sets over the fields $\mathbb{Z}_2, \mathbb{Z}_p$ (equipped with the usual modulo operations).

a) Let $f = x^2y + y^2x \in \mathbb{Z}_2[x, y]$. Compute $V(f) \subseteq \mathbb{Z}_2^2$, the algebraic set containing the zeros of f in $\mathbb{Z}_2^2$.

b) Find for arbitrary $n \in \mathbb{N}$ a polynomial $f \in \mathbb{Z}_2[x_1, \ldots, x_n]$ depending on all variables such that $V(f) = \mathbb{Z}_2$.

c) Find for arbitrary prime numbers p and $n \in \mathbb{N}$ a polynomial $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ depending on all variables such that $V(f) = \mathbb{Z}_p$.
Hint: For all $a \in \mathbb{Z}_p \setminus \{0\}$ it holds that $a^{p-1} = 1$ in $\mathbb{Z}_p$.

Exercise 8 A well-known theorem in geometry is the following: Every triangle ABC in $\mathbb{R}^2$ the lines orthogonal to the sides of the triangle, going through the midpoint of the corresponding side, have a point in common.



We want to prove this theorem by using Groebner bases:

a) Write f_1, f_2, f as polynomials in $\mathbb{R}[x, y]$ depending on the parameters a, b, c.

b) Show that $f \in \sqrt{\langle f_1, f_2 \rangle}$.

Exercise 9 [Lemma 2.2.8] Let I be an ideal in $K[x_1, \ldots, x_n]$ and let $p = (x_1 - a_1) \cdots (x_1 - a_d)$, where $a_1, \ldots, a_d$ are distinct. We want to show that

$$I + \langle p \rangle = \bigcap_{j=1}^{d} (I + \langle x_1 - a_j \rangle).$$

a) First show that $I + \langle p \rangle \subseteq \bigcap_j (I + \langle x_1 - a_j \rangle)$.

b) For the converse direction let $p_j = \prod_{i \neq j} (x_1 - a_i)$. Prove that $p_j \cdot (I + \langle x_1 - a_j \rangle) \subseteq I + \langle p \rangle$.

c) $p_1, \ldots, p_d$ are relatively prime, i.e. there are $h_1, \ldots, h_d$ such that $\sum_{j=1}^{d} h_j p_j = 1$. Show with this property and (b) that $\bigcap_{j=1}^{d} (I + \langle x_1 - a_j \rangle) \subseteq I + \langle p \rangle$.

Exercise 10 Let $I_1 = \langle x^2 - y^3 \rangle, I_2 = \langle x^2 - y, xy^2 \rangle$ and $I_3 = \langle x^2 y \rangle$.

a) Compute a basis for $I_4 = I_1 \cdot (I_2 \cap I_3), I_5 = I_1 + (I_2 \cap I_3)$ and $I_6 = I_4 : I_5$.

b) Visualize the corresponding algebraic sets of (a).

c) Check that I_2 is zero-dimensional and compute $\sqrt{I_2}$.

In the following examples we study the relation between a polynomial ideal

$$I = \langle f_1, \ldots, f_n \rangle \subseteq \mathbb{C}[x]$$

in one variable x and its corresponding algebraic set $V(I)$ without using Hilbert's Nullstellensatz.

Exercise 11 a) Find the generator $f \in \mathbb{C}[x]$ of I and compute $V(I)$. Show that the ideal vanishing on the common zeros of I, $I(V(I))$, is equal to $\langle f_{red} \rangle$, where f_{red} is the *square-free part* of f, i.e. in the factorization of f every factor occurs with multiplicity one.

b) Show that f and its derivative f' do not have a common factor if and only if $f = f_{red}$. Hint: Show that

$$f_{red} = \frac{f}{\gcd(f, f')}$$

holds.

c) Find f_{red} of

$$f = x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1$$

without factorization.

Exercise 12 a) Describe an algorithm whose input consists of polynomials $f_1, \ldots, f_n \in \mathbb{C}[x]$ and whose output consists of a basis of the ideal $I(V(f1, \ldots, f_n))$.

b) Find a basis for the ideal

$$I(V(x^5 - 2x^4 + 2x^2 - x,\ x^5 - x^4 - 2x^3 + 2x^2 + x - 1)).$$