Exercise sheet 1

meeting on **10/03/2020**

Exercise 1 Let

$$f_1 = y^2 - xy - xz + 1,$$
$$f_2 = 2xy - 2xz + x^2 + x,$$
$$f_3 = 2xy - 2xz + x^2 + 1.$$

Compute the common roots of f_1, f_2, f_3 both by resultants and by Groebner bases.

Exercise 2 Let

$$a = a_0 + a_1 x + a_2 x^2, \; b = b_0 + b_1 x$$

be polynomials in $\mathbb{Z}[x]$ with unknown coefficients.

a) Compute the polynomial relation between these coefficients if a and b have a common factor.

b) Let $a = 4x^2 - 3x + 1, \; b = 2x + 3$ and $I = \langle a, b\rangle \trianglelefteq \mathbb{Z}[x]$ be the ideal generated by a and b. Compute the ideal $I \cap \mathbb{Z} \trianglelefteq \mathbb{Z}$.

Exercise 3 Let $a, b \in k[x]$ and $\phi : k[x] \to k[x]$ be the following maps:

a) $\phi(a(x)) = a(x + c)$ for some $c \in k$;

b) $\phi(a(x)) = a(c\,x)$ for some $c \in k$;

c) $\phi(a(x)) = c(x)\,a(x)$ for some $c \in k[x]$.

Compute the resultant $Res_x(\phi(a), \phi(b))$ with respect to x for some specific c and try to find a relation for the general case.

Exercise 4 Let k be a field.

a) Let $f \in k[x_1, \ldots, x_n]$ such that $f \notin \langle x_1, \ldots, x_n\rangle$. Show that

$$\langle x_1, \ldots, x_n, f\rangle = k[x_1, \ldots, x_n].$$

b) Let $I \trianglelefteq k[x_1, \ldots, x_n]$ be a principal ideal (it is generated by one element $f \in k[x_1, \ldots, x_n]$). Show that any finite set $F \subset I$ containing f is a Groebner bases for I. Compute a normed reduced Groebner basis for I.

Exercise 5 Prove or disprove that the following rings are Noetherian:

a) Principal ideal rings such as the ring of integers $\mathbb{Z}$.

b) The Ring of polynomials in infinitely many variables such as $\mathbb{Q}[x_1, x_2, \ldots]$.

c) The ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 6 Let

$$f(x, y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2).$$

a) The curve implicitly defined by f can also be parametrized by

$$x = \frac{-1024t^3}{1 + 32t^2 + 256t^4}, \; y = \frac{128t^2 - 2048t^4}{1 + 32t^2 + 256t^4}.$$

Verify this by computing the elimination ideal as in Example 1.3 in the lecture notes.

b) Visualize the zeroset of f in $\mathbb{R}^2$ with its explicit and implicit description.