Commutative Algebra and Algebraic Geometry

Lecture Notes
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Contents

Preface

Commutative algebra is the theory of polynomial ideals. Problems concerning polynomial ideals are representation of ideals, e.g. by Gröbner bases or characteristic sets, decomposition of ideals into prime and primary ideals, determination of radical ideals, dimension and height of ideals, the structure of polynomial rings modulo ideals, solution of systems of polynomial equations, solution of linear systems with polynomial coefficients, i.e. computation of syzygies, etc. Constructive methods for answering such questions are provided by computer algebra, e.g. algorithms for computation of greatest common divisors, factorization, computation of Gröbner bases, resultants.

This theory of polynomial ideals can be generalized to vectors of polynomials, leading to modules over polynomial rings.

Algebraic geometry traditionally is the study of sets of solutions of systems of polynomial equations, i.e. of algebraic sets. We might be interested in the dimension of an algebraic set, its irreducible components, the tangent space at a point, functions on algebraic sets, different representations such as local or global parametrizations, etc.

Algebraic curves and surfaces are an old topic of geometric and algebraic investigation. They appear in biological shapes, in ancient and modern architectural designs, in number theoretic problems, in error-correcting codes, and in cryptographic algorithms. Recently they have gained additional practical importance as central objects in computer aided geometric design. Modern airplanes, cars, and household appliances would be unthinkable without the computational manipulation of algebraic curves and surfaces. Algebraic curves and surfaces combine fascinating mathematical beauty with challenging computational complexity and wide spread practical applicability.

Obviously commutative algebra and algebraic geometry are closely related. We will investigate some of these relations. For further reading on the topics of this course we suggest the books by Sendra, Winkler, Pérez-Díaz [SWP08], Cox, Little, O'Shea [CLO97], [CLO98], Fulton [Ful69], Reid [Rei88], and Walker [Wal50]. As good introductions to commutative algebra we suggest the books by Zariski, Samuel [ZaS58] and Kunz [Kun85]. For computational methods on polynomials we refer to the books by Winkler [Win96] and Kreuzer, Robbiano [KrR00]. Hartshorne [Har77] is a more advanced text on algebraic geometry. The modern language of algebraic geometry is introduced in the book by Eisenbud and Harris [EiH92].

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