Exam

Symbolic Linear Algebra

Summer Term 2017

Name: ______________________________

Matrikelnummer: ______________________________

SKZ: ______________________________

Problem	1	2	3	4	5	6	7	Total
Marks								
Maximum	1	1	2	2	2	6	6	20

Problem 1 (1 Mark). Show that $\mathbb{R}[x]$ is not a free $\mathbb{R}[\partial]$-module.

Problem 2 (1 Mark). Let R be a principal ideal domain and $a_1, \dots, a_n \in R$. Prove that all greatest common divisors of $a_1, \dots, a_n$ are associated.

Problem 3 (2 Marks). Let R be a principal ideal domain and let $v \in R^{n \times 1}$. Let $Q \in \mathrm{GL}_n(R)$ be such that

$$Qv = \begin{pmatrix} \gcd(v) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(a) Show that the first column of Q^{-1} is $v/\gcd(v)$.

(b) Prove that $\gcd(v/\gcd(v)) = 1$.

Problem 4 (2 Marks). Compute the Smith–Jacobson normal form of

$$A = \begin{pmatrix} x-1 & 0 & \cdots & 0 \\ 0 & x-2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x-n \end{pmatrix} \in \mathbb{R}[x]^{n \times n}.$$

Problem 5 (2 Marks). Let R be a principal ideal domain and $a, b \in R$. Show that the matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R^{2\times 2} \qquad \text{and} \qquad \begin{pmatrix} \gcd(a,b) & 0 \\ 0 & \operatorname{lcm}(a,b) \end{pmatrix} \in R^{2\times 2}$$

are equivalent.

Problem 6 (6 Marks). Use the Smith–Jacobson normal form to find all integer solutions of the linear diophantine system

$$\begin{aligned} -w + \ 4x + 4y - z &= 3, \\ w - \ 2x - 2y + z &= -1, \\ -w + \ 8x + 2y - z &= 1, \\ w - 10x - 4y + z &= -3. \end{aligned}$$

Problem 7 (6 Marks). Use the Hermite normal form to determine whether the matrix

$$A = \begin{pmatrix} x-1 & 1 & 1 \\ 2x^2 - x & x+1 & 2x \\ x-2 & 2 & 1 \end{pmatrix} \in \mathbb{R}[x]^{3\times 3}$$

is unimodular and compute the inverse in case that A is indeed unimodular.