Algorithmic Combinatorics

Exercises discussed on April 29, 2019

26. Use closure properties of the package GeneratingFunctions[1] to derive a recurrence for

$$s(n) = \sum_{k=0}^{n} L_{3k} F_{n-k},$$

where L_n are the Lucas numbers and F_n the Fibonacci numbers.

26. Given n people numbered from 1 to n sitting at a round table. Starting from person 1 in clockwise order every second person leaves until only one person remains (the first person to leave is person 2). Let $J(n)$ denote the number of the remaining person. Determine $J(n)$.

27. Characterize all sequences that are both C-finite and hypergeometric.

28. Determine the hypergeometric function representation of

(a) $\frac{1}{x}\log(1+x) = \sum_{n\geq 0} \frac{(-1)^n}{n+1} x^n$

(b) $\cos(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} x^{2n}$

(c) $\frac{1}{x}\arctan(x) = \sum_{n\geq 0} \frac{(-1)^n}{2n+1} x^{2n}$

29. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have the hypergeometric series representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n \quad n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right).$$

Show that the derivative of Jacobi polynomials is again a Jacobi polynomial with shifted parameters, i.e., show that

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Chebyshev polynomials of the first kind $T_n(x)$ are special instances of Jacobi polynomials. Which parameters α, β do they correspond to?

[1] available at https://combinatorics.risc.jku.at/software