Algorithmic Combinatorics
Exercises discussed on April 8, 2019

22. Let the sequence $\big(f(n)\big)_{n\geq 0}$ be recursively defined by

$$f_{n+3} + 2f_{n+2} - f_{n+1} - 2f_n = 0, \qquad f_0 = 1,\ f_1 = -1,\ f_2 = 2.$$

What is the companion matrix of this recurrence? Determine the solution of the recurrence analogously to the Fibonacci example presented in the lecture.

23. Prove Theorem 3.2 for recurrences of order two, i.e., for $c_0 \neq 0, c_1 \in \mathbb{K}$ with

$$x^2 + c_1 x + c_0 = (x - \alpha_1)(x - \alpha_2)$$

with $\alpha_1 \neq \alpha_2 \in \mathbb{K}$, show that $(\alpha_1^n)_{n\geq 0}, (\alpha_2^n)_{n\geq 0}$ form a basis for the solutions of the recurrence

$$a_{n+2} + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

and that if $\alpha_1 = \alpha_2 = \alpha$ then $(\alpha^n)_{n\geq 0}, (n\alpha^n)_{n\geq 0}$ form a basis for the solutions of the above recurrence.

24. (Theorem 3.3) Show that a sequence $(a_n)_{n\geq 0}$ in $\mathbb{K}$ satisfies a C-finite recurrence

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

with $c_i \in \mathbb{K}, c_0 \neq 0$, if and only if

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0 x^r}$$

for some polynomial $p(x)$ with degree at most $r-1$.

25. (Tower of Hanoi) Given a tower of n disks initially stacked in increasing order on one of three pegs, the task is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger disk onto a smaller one. Let a_n denote the minimal number of moves needed.

Find a recurrence for a_n. Compute the first few values and guess a closed form solution. Derive the closed form solution using techniques from the lecture.