In[∘]:=
```
<< RISC`GeneratingFunctions`
<< RISC`HolonomicFunctions`
<< RISC`Guess`
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Guess Package version 0.52
written by Manuel Kauers

In[∘]:=
```
fibonacci = {a[n + 2] - a[n + 1] - a[n] == 0, a[0] == 0, a[1] == 1};
lucas = {a[n + 2] - a[n + 1] - a[n] == 0, a[0] == 2, a[1] == 1};
```

In[∘]:=
```
(* s[n] = Sum[LucasL[3k] Fibonacci[n-k],{k,0,n}] - HW 6/26 *)
```

In[∘]:=
```
? RESubsequence
```

RecurrenceEquationSubsequence[re,a[n],m*n+k] gives a recurrence that
is satisfied by a subsequence of the form a[m*n+k] of every solution a[n] of
the input recurrence re.

Alias: RESubsequence

See also: REInfo, REInterlace

In[∘]:=
```
lucas3 = RESubsequence[lucas, a[n], 3 n]
```

Out[∘]=
```
{-a[n] - 4 a[1 + n] + a[2 + n] == 0, a[0] == 2, a[1] == 4}
```

In[∘]:=
```
RECauchy[lucas3, fibonacci, a[n]]
```

Out[∘]=
```
{a[n] + 5 a[1 + n] + 2 a[2 + n] - 5 a[3 + n] + a[4 + n] == 0,
 a[0] == 0, a[1] == 2, a[2] == 6, a[3] == 26}
```

In[•]:=
```
data = Table[Sum[LucasL[3 k] Fibonacci[n - k], {k, 0, n}], {n, 0, 20}]
```
Out[•]=
```
{0, 2, 6, 26, 108, 456, 1928, 8162, 34 566, 146 410, 620 180,
 2 627 088, 11 128 464, 47 140 834, 199 691 622, 845 907 034, 3 583 319 292,
 15 179 183 448, 64 300 051 864, 272 379 388 930, 1 153 817 604 390}
```

In[•]:=
```
GuessRE[data, a[n]]
```
Out[•]=
```
FAIL
```

In[•]:=
```
? GuessRE
```

In[•]:=
```
GuessRE[data, a[n], {1, 4}, {0, 5}]
```
Out[•]=
```
{{a[n] + 5 a[1 + n] + 2 a[2 + n] - 5 a[3 + n] + a[4 + n] == 0,
  a[0] == 0, a[1] == 2, a[2] == 6, a[3] == 26}, ogf}
```

In[•]:=
```
? Guess*
```

▼ RISC`GeneratingFunctions`

Guess	GuessDE	GuessRationalFunction
GuessAE	GuessDifferentialEquation	GuessRE
GuessAlgebraicEquation	GuessRatF	GuessRecurrenceEquation

▼ RISC`Guess`

GuessCurveRE	GuessMinDE	GuessMultDE	GuessSymmetr-icRE	GuessUnivDE
GuessMinAE	GuessMinRE	GuessMultRE	GuessUnivAE	GuessUnivRE

In[•]:=
```
GuessMinRE[data, a[n], Infolevel → 3]
```
```
1 solutions predicted.
Q.
9 223 372 036 854 775 783
9 223 372 036 854 775 643
{0.000139, 0, 0.000238, 0.000805}
```
Out[•]=
```
a[n] + 5 a[1 + n] + 2 a[2 + n] - 5 a[3 + n] + a[4 + n]
```

In[•]:=
```
ann = Annihilator[Sum[LucasL[3 k] Fibonacci[n - k], {k, 0, n}], S[n]]
```
Out[•]= $\left\{S_n^4 - 5\,S_n^3 + 2\,S_n^2 + 5\,S_n + 1\right\}$

In[•]:=
```
ApplyOreOperator[ann, a[n]]
```
Out[•]=
```
{a[n] + 5 a[1 + n] + 2 a[2 + n] - 5 a[3 + n] + a[4 + n]}
```

```
In[•]:= ApplyOreOperator[Sum[LucasL[3 k] Fibonacci[n - k], {k, 0, n}]]

Out[•]= ApplyOreOperator[
  -((2^(2-3 n) (-1 - Sqrt[5])^(-1-3 n) ((-4)^n (1 + Sqrt[5])^(1+4 n) + 64^n (5 + 2 Sqrt[5]) + (ⅈ (1 + Sqrt[5]))^(6 n) (5 + 2 Sqrt[5]) -
        16^n (1 + Sqrt[5])^(2 n) (11 + 5 Sqrt[5]))) / (3 Sqrt[5] (3 + Sqrt[5])))]

In[•]:= Sum[LucasL[3 k] Fibonacci[n - k], {k, 0, n}]

Out[•]= -((2^(2-3 n) (-1 - Sqrt[5])^(-1-3 n) ((-4)^n (1 + Sqrt[5])^(1+4 n) + 64^n (5 + 2 Sqrt[5]) +
        (ⅈ (1 + Sqrt[5]))^(6 n) (5 + 2 Sqrt[5]) - 16^n (1 + Sqrt[5])^(2 n) (11 + 5 Sqrt[5]))) / (3 Sqrt[5] (3 + Sqrt[5])))

In[•]:= ? Annihilator

In[•]:= Sum[k ! HarmonicNumber[n - k], {k, 0, n}]

Out[•]= Sum[k ! HarmonicNumber[-k + n], {k, 0, n}]

In[•]:= factorial = {a[n + 1] - (n + 1) a[n] == 0, a[0] == 1}

Out[•]= {-(1 + n) a[n] + a[1 + n] == 0, a[0] == 1}

In[•]:= harmonic = {ApplyOreOperator[
     First[Annihilator[HarmonicNumber[n], S[n]]], a[n]], a[0] == 0, a[1] == 1}

Out[•]= {(1 + n) a[n] + (-3 - 2 n) a[1 + n] + (2 + n) a[2 + n], a[0] == 0, a[1] == 1}

In[•]:= cp = RECauchy[factorial, harmonic, a[n]]

   Solve: Equations may not give solutions for all "solve" variables.

   Solve: Equations may not give solutions for all "solve" variables.

Out[•]= {4 (2 + n)^3 (4 + n) a[n] - 4 (3 + n) (130 + 124 n + 39 n^2 + 4 n^3) a[1 + n] +
     (4 + n) (1512 + 1172 n + 300 n^2 + 25 n^3) a[2 + n] +
     (-10 606 - 8961 n - 2802 n^2 - 382 n^3 - 19 n^4) a[3 + n] +
     (9284 + 6524 n + 1669 n^2 + 182 n^3 + 7 n^4) a[4 + n] +
     (-3972 - 2284 n - 463 n^2 - 38 n^3 - n^4) a[5 + n] +
     (6 + n) (120 + 33 n + 2 n^2) a[6 + n] - (6 + n) (7 + n) a[7 + n] == 0,
   a[0] == 0, a[1] == 1, a[2] == 5/2, a[3] == 16/3, a[4] == 155/12,
   a[5] == 1231/30, a[6] == 1759/10}

In[•]:= data = Table[Sum[k ! HarmonicNumber[n - k], {k, 0, n}], {n, 0, 40}];

In[•]:= GuessMinRE[data, a[n]]

Out[•]= (-16 - 20 n - 8 n^2 - n^3) a[n] + (105 + 98 n + 30 n^2 + 3 n^3) a[1 + n] +
   (-212 - 157 n - 38 n^2 - 3 n^3) a[2 + n] + (172 + 99 n + 18 n^2 + n^3) a[3 + n] +
   (-54 - 21 n - 2 n^2) a[4 + n] + (5 + n) a[5 + n]
```

In[•]:=
```
guess = GuessRE[data, a[n], 5, 3][[1]]
```
Out[•]=
```
{-(2 + n)^2 (4 + n) a[n] + (3 + n) (35 + 21 n + 3 n^2) a[1 + n] - (4 + n) (53 + 26 n + 3 n^2) a[2 + n] +
    (4 + n) (43 + 14 n + n^2) a[3 + n] - (6 + n) (9 + 2 n) a[4 + n] + (5 + n) a[5 + n] == 0,
  a[0] == 0, a[1] == 1, a[2] == 5/2, a[3] == 16/3, a[4] == 155/12}
```

In[•]:=
```
guess2 =
  {-(2 + n)^2 (4 + n) a[n] + (3 + n) (35 + 21 n + 3 n^2) a[1 + n] - (4 + n) (53 + 26 n + 3 n^2) a[2 + n] +
      (4 + n) (43 + 14 n + n^2) a[3 + n] - (6 + n) (9 + 2 n) a[4 + n] + (5 + n) a[5 + n] == 0,
    a[0] == 0, a[1] == -1, a[2] == -5/2, a[3] == -16/3, a[4] == -155/12}
```
Out[•]=
```
{-(2 + n)^2 (4 + n) a[n] + (3 + n) (35 + 21 n + 3 n^2) a[1 + n] - (4 + n) (53 + 26 n + 3 n^2) a[2 + n] +
    (4 + n) (43 + 14 n + n^2) a[3 + n] - (6 + n) (9 + 2 n) a[4 + n] + (5 + n) a[5 + n] == 0,
  a[0] == 0, a[1] == -1, a[2] == -5/2, a[3] == -16/3, a[4] == -155/12}
```

In[•]:=
```
REPlus[cp, guess2, a[n]]
```
Out[•]=
```
{-4 (2 + n)^3 (4 + n) a[n] + 4 (3 + n) (130 + 124 n + 39 n^2 + 4 n^3) a[1 + n] -
    (4 + n) (1512 + 1172 n + 300 n^2 + 25 n^3) a[2 + n] +
    (10 606 + 8961 n + 2802 n^2 + 382 n^3 + 19 n^4) a[3 + n] +
    (-9284 - 6524 n - 1669 n^2 - 182 n^3 - 7 n^4) a[4 + n] +
    (3972 + 2284 n + 463 n^2 + 38 n^3 + n^4) a[5 + n] -
    (6 + n) (120 + 33 n + 2 n^2) a[6 + n] + (6 + n) (7 + n) a[7 + n] == 0,
  a[0] == 0, a[1] == 0, a[2] == 0, a[3] == 0, a[4] == 0, a[5] == 0, a[6] == 0}
```

In[•]:=
```
cp
```
Out[•]=
```
{4 (2 + n)^3 (4 + n) a[n] - 4 (3 + n) (130 + 124 n + 39 n^2 + 4 n^3) a[1 + n] +
    (4 + n) (1512 + 1172 n + 300 n^2 + 25 n^3) a[2 + n] +
    (-10 606 - 8961 n - 2802 n^2 - 382 n^3 - 19 n^4) a[3 + n] +
    (9284 + 6524 n + 1669 n^2 + 182 n^3 + 7 n^4) a[4 + n] +
    (-3972 - 2284 n - 463 n^2 - 38 n^3 - n^4) a[5 + n] +
    (6 + n) (120 + 33 n + 2 n^2) a[6 + n] - (6 + n) (7 + n) a[7 + n] == 0,
  a[0] == 0, a[1] == 1, a[2] == 5/2, a[3] == 16/3, a[4] == 155/12,
  a[5] == 1231/30, a[6] == 1759/10}
```

```
(* Catalan *)
```

In[•]:=
```
AE2DE[x f[x]^2 - f[x] + 1 == 0, f[x]]
```
Out[•]=
```
-1 - (-1 + 2 x) f[x] - (-x + 4 x^2) f'[x] == 0
```

In[•]:=
```
AE2DE[{x f[x]^2 - f[x] + 1 == 0, f[0] == 1}, f[x]]
```
Out[•]=
```
{-1 - (-1 + 2 x) f[x] - (-x + 4 x^2) f'[x] == 0, f[0] == 1}
```

In[•]:=
```
DE2RE[AE2DE[{x f[x]^2 - f[x] + 1 == 0, f[0] == 1}, f[x]], f[x], a[n]]
```
Out[•]= $\{2\,(1+n)\,(1+2\,n)\,a[n]-(1+n)\,(2+n)\,a[1+n]==0,\ a[0]==1\}$

In[•]:=
```
Annihilator[LegendreP[n, x], S[n]]
```
Out[•]= $\{(2+n)\,S_n^2+(-3\,x-2\,n\,x)\,S_n+(1+n)\}$

In[•]:=
```
Annihilator[LegendreP[n, x], {S[n], Der[x]}]
```
Out[•]= $\{(1+n)\,S_n+(1-x^2)\,D_x+(-x-n\,x),\ (-1+x^2)\,D_x^2+2\,x\,D_x+(-n-n^2)\}$

In[•]:=
```
Annihilator[LegendreP[n, x], {Der[x], S[n]}]
```
Out[•]= $\{(1-x^2)\,D_x+(1+n)\,S_n+(-x-n\,x),\ (2+n)\,S_n^2+(-3\,x-2\,n\,x)\,S_n+(1+n)\}$

In[•]:=
```
Annihilator[Binomial[n, k], {S[n], S[k]}]
```
Out[•]= $\{(1+k)\,S_k+(k-n),\ (1-k+n)\,S_n+(-1-n)\}$

In[•]:=
```
Annihilator[StirlingS2[n, k], {S[n], S[k]}]
```
Annihilator: The expression StirlingS2[n, k] is not recognized to be ∂–finite. The result might not generate a zero–dimensional ideal.

Out[•]= $\{S_n\,S_k+(-1-k)\,S_k-1\}$

In[•]:=
```
legendre = {ApplyOreOperator[Annihilator[LegendreP[n, x], S[n]][[1]], p[n]] == 0,
   p[0] == LegendreP[0, x], p[1] == LegendreP[1, x]}
```
Out[•]= $\{(1+n)\,p[n]+(-3\,x-2\,n\,x)\,p[1+n]+(2+n)\,p[2+n]==0,\ p[0]==1,\ p[1]==x\}$

In[•]:=
```
ode = RE2DE[legendre, p[n], F[z]]
```
Out[•]= $\{-(x-z)\,F[z]-(-1+2\,x\,z-z^2)\,F'[z]==0,\ F[0]==1\}$

In[•]:=
```
DSolve[ode, F[z], z]
```
Out[•]= $\left\{\left\{F[z]\to\frac{1}{\sqrt{1-2\,x\,z+z^2}}\right\}\right\}$

In[•]:=
```
<< RISC`fastZeil`
```

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[•]:=
```
? Gosper
```

Gosper[function, range],
uses Gosper's algorithm to find a hypergeometric closed form for the sum
of the function over the range,

Gosper[function, k], computes the hypergeometric forward anti–difference
of function in k, if it exists,

Gosper[function, range, degree] or

Gosper[function, k, degree] use Gosper's algorithm with an
undetermined polynomial of given degree in k multiplied to the function.

In[•]:=
```
Gosper[k k!, k]
```

Out[•]= $\{k\, k! == \Delta_k[k!]\}$

In[•]:=
```
Gosper[k k!, {k, 0, n}]
```

```
If `n' is a natural number, then:
```

Out[•]= {Sum[k k!, {k, 0, n}] == -1 + (1 + n) n!}

In[•]:=
```
Gosper[k!, k]
```

Out[•]= { }

In[•]:=
```
Gosper[Binomial[2 k, k] a^k, {k, 0, n}]
```

Out[•]= { }

In[•]:=
```
Gosper[Binomial[2 k, k] (1 / 4)^k, {k, 0, n}]
```

```
If `n' is a natural number, then:
```

Out[•]= $\{\mathrm{Sum}[4^{-k}\,\mathrm{Binomial}[2k, k], \{k, 0, n\}] == 4^{-n}\,(1 + 2n)\,\mathrm{Binomial}[2n, n]\}$

In[•]:=
```
Gosper[Binomial[x + k, k], {k, 0, n}]
```

If `n' is a natural number and $1 + x \neq 0$, then:

Out[•]= $\{\mathrm{Sum}[\mathrm{Binomial}[k + x, k], \{k, 0, n\}] == \frac{1}{1 + x}(1 + n + x)\,\mathrm{Binomial}[n + x, n]\}$

In[•]:=
```
Gosper[Binomial[n, k], {k, 0, n}]
```

Out[•]= { }

In[∘]:=
```
Zb[Binomial[n, k], {k, 0, n}, n]
```

```
If `n' is a natural number, then:
```

Out[∘]= $\{2\,\mathrm{SUM}[n] - \mathrm{SUM}[1+n] == 0\}$

In[∘]:=
```
<< Hyper.m
```

In[∘]:=
```
Hyper[(n - 1) y[n + 2] - (3 n - 2) y[n + 1] + 2 n y[n], y[n], Solutions → All]
```

Out[∘]= $\left\{2, \frac{1+n}{n}\right\}$