Ad (6) $\deg b_0 < r$

Ex: $x^2 y' + y = 0$ cannot have a series solution of the form $y = x^\nu + \dots$, because

$$x^2(\nu x^{\nu-1} + \dots) + (x^\nu + \dots) = 1 \cdot x^\nu + \dots \neq 0.$$

The degree increase caused by x^2 exceeds the degree drop caused by differentiation. The recurrence here is $1 \cdot y_n + (n-1) y_{n-1}$. There is no index n where a coefficient sequence could possibly start.

Idea: Apply substitutions that lead to nontrivial indicial equations.

We will use the differentiation rule

$$D_x\, e^{cx^k} = c \cdot k \cdot x^{k-1} e^{cx^k} \qquad (c \in \mathbb{C}, k \in \mathbb{Q})$$

to influence the degrees of the polynomial coeffs of the given DEQ so as to induce nontrivial indicial equations.

For $m \geq 0$, let the polynomials $e_m \in \mathbb{Q}[k,z]$ be defined ~~so~~ through

$$D_x^m e^{cx^k} = e_m(k, cx^k) \cdot x^{-m} e^{cx^k}$$

By induction, it is easy to see that

$$\deg_z e_m = m \quad \text{and} \quad [z^m]\, e_m = k^m.$$

Therefore, if $k < 0$, the smallest (!) exponent of x in $e_m(k, cx^k)$ is mk.

Now consider the DEQ

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0.$$

Let $d_0, \dots, d_r$ be the minimal exponents in $a_0 \dots a_r$, respectively. ($d_i := \infty$ if $a_i = 0$ for some i.) Set $y = e^{cx^k} \tilde{y}$ for as yet undetermined $c, k, \tilde{y}$. Then using

$$D^i(y) = \sum_j \binom{i}{j} D^{i-j}(e^{cx^k}) D^j(\tilde{y})$$

$$= \sum_j \binom{i}{j} e_{i-j}(k, cx^k)\, x^{-(i-j)} D^j(\tilde{y}) \cdot e^{cx^k}$$

the DEQ for y turns into

$$\sum_i a_i \, D^i(y) = 0$$

$$\sum_i a_i \sum_j \binom{i}{j} e_{i-j}(k, cx^k) \, x^{-(i-j)} \, D^j(y) = 0$$

$$\sum_j \Big(\underbrace{\sum_i a_i \binom{i}{j} e_{i-j}(k, cx^k) \, x^{-(i-j)}} \Big) D^j(y) = 0$$

$$= \underbrace{([x^{d_i}]a_i) \binom{i}{j} (kc)^{i-j}}_{\neq 0 \text{ if } k<0 \text{ and } i \geq j.} \; x^{d_i + (i-j)(k-1)} + \dots$$

want: values of k such that the lowest order term of the inner sum is determined by more than one summand, viz that none of the values

$$d_i + (i-j)(k-1) \qquad (i = j..r)$$

is smaller than all the others, viz. that

$$\min_{i=j}^{r} \big(d_i + (i-j)(k-1) \big)$$

is attained for at least two different indices i_1, i_2.

Observe that the desired property is not affected if ~~all~~ we ~~add~~ $j \cdot (k-1)$ to all the numbers.

Hence the problem is independent of j.
It suffices to consider $d_i + i(k-1)$ $(i = 0..r)$.

Brute force: solving

$$d_{i_1} + i_1(k-1) = d_{i_2} + i_2(k-1)$$

$$k = 1 - \frac{d_{i_1} - d_{i_2}}{i_1 - i_2}$$

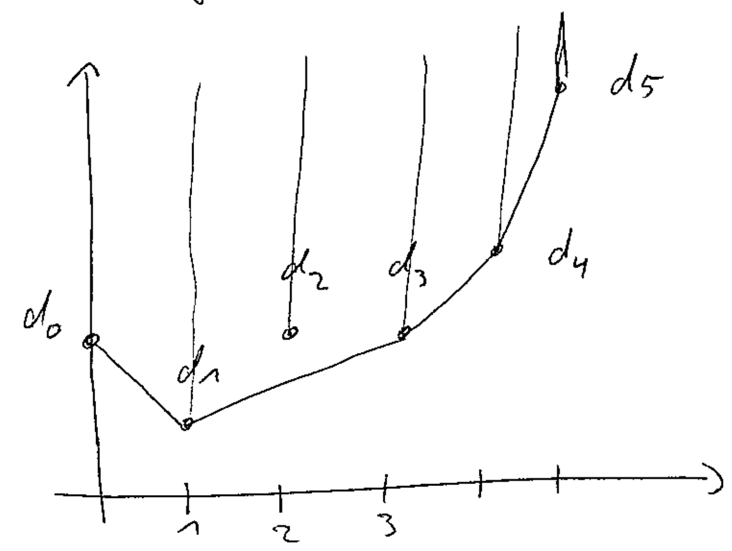
for $0 \le i_1 < i_2 \le r$ gives finitely many candidates for k. For each of them, check whether

(a) $k < 0$

(b) $\nexists\, i_3 \in \{0 \ldots r\} : d_{i_3} + i_3(k-1) < d_{i_1} + i_1(k-1)$

These are the eligible values for k.

Geometrically:



The Newton-polygon of a DEQ is defined as the convex hull of all the halflines $\{(i, y) \mid y \geq d_i\}$ $(i = 0..r)$. Eligible values for k are precisely the values $1-\alpha$ where $\alpha > 1$ is the slope of some edge in this polygon.

For each such k, forcing the lowest order term in the equation for $\tilde{y}$ to 0 gives a nontrivial polynomial equation for c. Replacing c by some solution (in $\bar{C}$) of this equation gives a new differential equation for $\tilde{y}$.

This new equation may or may not have a nontrivial indicial equation. If it has not, the process is repeated until an equation with a nontrivial indicial equation appears.

In the end, if $\tilde{y} = x^{\nu} (\log x)^{m} u$ is a generalized series solution of a transformed equation, it translates to a solution

$$y = e^{c_1 x^{k_1} + c_2 x^{k_2} + \dots + c_e x^{k_e}} \cdot x^{\nu} (\log x)^{m} u$$

of the original equation.
(We will enforce $k_1 < k_2 < \dots < k_e$ to avoid getting the same solutions twice.)

In summary, the algorithm works as follows.

<u>Alg.</u>

Input: A DEQ $a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0$
a bound B (initially $B := -\infty$)

Output: generalized series solutions
$$y = e^{c_1 x^{k_1} + \dots + c_e x^{k_e}} x^{\nu} (\log x)^m u$$
with $B < k_1 < \dots < k_e < 0$,
$c_1 \dots c_e \in \bar{C} \setminus \{0\}$, $\nu \in \bar{C}$, $m \in \mathbb{N}$,
$u \in \bar{C}[[x]]$ (truncated)

(1) Construct the Newton Polygon of the DEQ.

(2) $S = \emptyset$

(3) for all eligible k with $B < k$ do

(4) determine the corresponding values $c \in \bar{C}$ as described above

(5) for each such c do

(6) plug $y = e^{cx^k} \bar{y}$ into the eq and apply the alg recursively to the resulting eq and $B = k$. Let $\tilde{S}$ be the output.

(7) $S = S \cup \{ e^{cx^k} \tilde{y} \mid \tilde{y} \in \tilde{S} \}$

(8) determine all solutions of the form $y = x^{\nu} (\log x)^m u$ and add them to S

(9) return S.

It can be shown that this algorithm terminates and returns a fundamental system of the input DEQ.

(4) Hyperexponential Solutions

<u>Task:</u> Given $a_0, \dots, a_r \in C[x]$, find all y hyperexponential over $C(x)$ with
$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0.$$

<u>Recall:</u> If E is a differential field containing $C(x)$, and if $y \in E$, then y is called hyperexponential (hexp) over $C(x)$ if $D(y) = u\,y$ for some $u \in C(x)$.

<u>Lemma:</u> Each hyperexponential term y over $C(x)$ can be written
$$y = p_1^{e_1} \cdots p_n^{e_n} e^{v}$$
for some $e_1 \dots e_n \in \overline{C}$, $p_1 \dots p_n \in \overline{C}[x]$, $v \in C(x)$.
More precisely: For every $u \in C(x)$ there exists an extension $E = C(x)(p_1^{e_1}, \dots, p_n^{e_n})(e^{v})$ of $C(x)$ and an element $y = p_1^{e_1} \cdots p_n^{e_n} e^{v} \in E$ with $D(y) = u \cdot y$.

Proof: First note that

$$\frac{D(p_1^{e_1}\cdots p_n^{e_n} e^v)}{p_1^{e_1}\cdots p_n^{e_n} e^v} = \frac{\left(\sum_i e_i p_i^{e_i-1} D(p_i) \prod_{j\neq i} p_j^{e_j}\right)\cdot e^v + \prod_i p_i^{e_i} e^v D(v)}{p_1^{e_1}\cdots p_n^{e_n} e^v}$$

$$= \sum_i e_i \frac{D(p_i)}{p_i} + D(v).$$

Next observe that every $u \in C(x)$ can be written in this form (rational integration)

$\boxtimes$

Poles of v are called irregular singularities of y, and y is said to have an irregular singularity "at ∞" if $\deg_x \mathrm{num}(v) > \deg_x \mathrm{denom}(v)$.

Roots of p_i with $e_i \notin \mathbb{N}$ are called regular singularities (unless they are also roots of $\mathrm{denom}(v)$), and y is said to have a regular singularity at ∞ if $e_i \in \mathbb{N}$ for some i and it has no irregular singularity at ∞.

If y is a hyperexponential solution of an ODE, then each of its finite singularities must be a root of the ODE's leading coefficient. Singularities at ∞ turn into singularities at 0 by the change of variables $\tilde{x} = \frac{1}{x}$.

Idea behind the following algorithm:
A necessary (and sufficient!) condition for y to be a solution of a given DEQ is that its series expansions in all the singularities are solutions of the DEQ.

<u>Alg</u>. (sketch)
Input: A DEQ $a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0$
Output: all the hyperexponential solutions of this DEQ.

(1) $E = \{1\}$

(2) Let $c_0 = \infty$ and $c_1, \dots, c_\ell \in \bar{C}$ be the roots of a_r.

(3) for $i = 0 \dots \ell$ do

(4) obtain $\widetilde{DEQ}$ from DEQ by substituting

$$\tilde{x} = \begin{cases} 1/x & \text{if } i = 0 \\ x - c_i & \text{if } i > 0. \end{cases}$$

(5) use the previous alg to compute the set S of all $p \in \bar{C}[\tilde{x}]$ such that $\widetilde{DEQ}$ has a solution $e^{p(1/\tilde{x})} \cdot x^v (\log x)^m u$.

(6) $E = \{ A \cdot e^{p(1/\tilde{x})} \mid A \in S,\ p \in S \}$

(7) $S = \emptyset$

(8) for each $e^v \in E$ do

(9) obtain $\widetilde{DEQ}$ from DEQ by substituting $y = e^v \tilde{y}$ for unknown $\tilde{y}$.

(10) compute the solutions $p_1^{e_1} \cdots p_n^{e_n}$ of $\widetilde{DEQ}$ using the algorithm for finding rational solutions, but without the restriction that $e_1 \dots e_n$ should be integers

(11) for each such $p_1^{e_1} \cdots p_n^{e_n}$,
set $S := S \cup \{ p_1^{e_1} \cdots p_n^{e_n} \cdot e^v \}$

(12) return S.