② Rational Solutions

Task: Given $a_0, \dots, a_r \in C[x]$, find all $y \in C(x)$ st

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0,$$

where $D = \frac{d}{dx}$ is the standard derivation.

Suppose $y = \frac{p}{q}$ is a solution, and suppose we know some $Q \in C[x]$ with $q \mid Q$. Then we can write $y = \frac{p}{q} = \frac{P}{Q}$ for some (unknown) $P \in C[x]$ (not nec coprime with Q). Plugging this into the equation gives

$$\tilde{a}_0 P + \tilde{a}_1 D(P) + \dots + \tilde{a}_r D^r(P) = 0$$

for certain $\tilde{a}_i \in C(x)$. P is a polynomial solution of this equation $\iff$ $\frac{P}{Q}$ is a rational solution of the original equation.

Since the solution space is finite dimensional over C, there must be some $Q \in C[x]$ such that $\mathrm{denom}(y) \mid Q$ for every rational solution $y \in C(x)$.

How to find it?

Let $y = \frac{p}{q}$ be a solution and $q = q_1^{e_1} \cdots q_n^{e_n}$ be the factorization of q over $C[x]$.

Then

$$D^i\left(\frac{p}{q}\right) = \frac{\text{poly}}{q_1^{e_1+i} \cdots q_n^{e_n+i}}$$

without the possibility of a cancelation

(Proof: $$D\left(\frac{u}{q^e v}\right) = \frac{D(u) q^e v - u\, D(q^e v)}{q^{2e} v^2}$$

$$= \frac{D(u)\, q v - u e\, D(q) v - u q\, D(v)}{q^{e+1} v^2}$$

$q \mid$ numerator $\iff$ $q \mid u e\, D(q) v$, which is not the case because $e > 0$, $\gcd(q, u) = \gcd(q, D(q)) = \gcd(q, v) = 1$.)

Therefore, in order to have

$$a_0 y + a_1 D(y) + \cdots + a_r D^r(y) = 0$$

viz. $$\underbrace{a_r D^r(y)}_{\frac{\text{poly}}{q_1^{e_1+r} \cdots q_n^{e_n+r}}} = \underbrace{-a_0 y - \cdots - a_{r-1} D^{r-1}(y)}_{= \frac{\text{poly}}{q_1^{e_1+r-1} \cdots q_n^{e_n+r-1}}}$$

we must have a cancelation on the lhs. Since no cancelation with numerator $(D^r(y))$ is possible, it follows $q_1 \cdots q_n \mid a_r$.

Hence: An irreducible factor of the denominator of a solution y must also be a factor of a_r.

Todo: Bounds on the multiplicities.

Given an irred factor $q \in \mathbb{C}[x]$ of a_r, we want to find $E \in \mathbb{Z}$ such that for every solution $y = \frac{u}{q^e v}$ with $\gcd(q,u) = \gcd(q,v) = \gcd(u,v) = 1$ we have $e \geq E$.

Note: $$D^i\left(\frac{u}{q^e v}\right) = \frac{q\,\text{poly} - e^{\underline{i}}\, D^i(q)\, u v}{q^{e+i}\, v^{2(i+1)}} \qquad (i>0)$$

Therefore, if d_i denotes the multiplicity of q in a_i $(i = 0..r)$, we have

$$a_i\, D^i\left(\frac{u}{q^e v}\right) = \frac{q\, A_i - e^{\underline{i}}\, B_i}{q^{e+i-d_i}\, v^{2(i+1)}} \qquad (i>0)$$

for certain $A_i, B_i \in \mathbb{C}[x]$ with $q \nmid B_i$

and $a_0\, D^0\left(\frac{u}{q^e v}\right) = \frac{B_0}{q^{e-d_0}\, v}$.

for some $B_0 \in \mathbb{C}[x]$ with $q \nmid B_0$.

(set $A_0 = 0$)

Let $m := \min_{i=0}^{r} (d - d_i)$. Then

$$q^{e+m} \sum_{i=0}^{r} a_i D^i\left(\frac{u}{q^e v}\right) = \sum_{i=0}^{r} q^{m-(d-d_i)} \left(q A_i - e^{\underline{i}} B_i\right)$$

$$\equiv \sum_{i=0}^{r} e^{\underline{i}} C_i \mod q$$

for certain $C_i \in C[x]$ of degree $< \deg q$.
By the choice of m, and because $q \nmid B_i$ for all i, not all the C_i can be zero. Comparing coefficients wrt x ~~and~~ gives some nonzero polynomial equations for e. Solving them simultaneously for e gives at most finitely many solutions. The smallest of them qualifies for $\mathcal{E}$.

(3) Series Solutions

Task: Given $a_0 \ldots a_r \in C[x]$, find $y \in C[[x]]$ (up to a prescribed precision) such that

$$a_0 y + a_1 D(y) + \cdots + a_r D^r(y) = 0.$$

(again $D = \frac{d}{dx}$).

If $y = \sum_n y_n x^n$, then $D(y) = \sum_n y_n x^{n-1} = \sum_n y_{n+1}(n+1)$

$D^2(y) = \sum_n y_{n+2}(n+2)(n+1)x^n$

etc

Substituting these expressions into the equation and picking the coefficient of x^n yields a linear recurrence equation for y_n with coefficients depending polynomially on n.

If the DE has order r and degree d, the resulting RE has order $r+d$ and degree r.

Ex! $y' - y = 0$ $\quad y = \sum_n y_n x^n \quad y' = \sum_n y_{n+1}(n+1)x^n$

$$\sum_n (y_{n+1}(n+1) - y_n)x^n = 0$$

Rec: $(n+1)y_{n+1} = y_n \rightsquigarrow y_n = \frac{1}{n!}$

Note: The recurrence holds for all $n \in \mathbb{Z}$.

~~Therefore~~ If y is a power series solution with $\operatorname{ord}(y) = \nu \in \mathbb{N}$, we have

$$y_{\nu-1} = y_{\nu-2} = y_{\nu-3} = \cdots = y_{\nu-m} = \cdots = 0$$

and the recurrence

$$b_0(n) y_n + b_{-1}(n) y_{n-1} + \dots + b_{-m}(n) y_{n-m} = 0$$

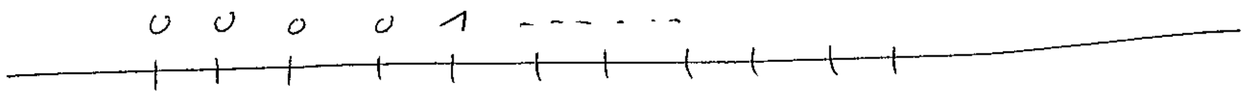
applied to $n = \nu$ yields

$$\underbrace{b_0(\nu) \underbrace{y_\nu}_{\neq 0}}_{} + \underbrace{b_{-1}(\nu) \underbrace{y_{\nu-1}}_{=0} + \dots + b_{-m}(\nu) \underbrace{y_{\nu-m}}_{=0}}_{=0} = 0$$

$\Rightarrow \; b_0(\nu) = 0$

Hence: If the DE has a power series solution of order $\nu \in \mathbb{N}$ then $b_0(\nu) = 0$.

Conversely, if $\nu \in \mathbb{N}$ is a root of b_0 such that $b_0(\nu + k + 1) \neq 0$ for all $k \in \mathbb{N}$, then the DE has a power series solution of order ν.



(The value of y_ν is irrelevant and can be chosen $= 1$ because the solutions form a $\mathbb{C}$-vector space)

More generally, if $\nu \in \mathbb{C}$ is any root of b_0 such that $b_0(\nu + k) \neq 0$ $\forall k \in \mathbb{N}$ ($\nu \notin \mathbb{N}$ allowed), we find a generalized series solution of the form

$$y = x^{\nu} \cdot u(x)$$

for some $u \in \mathbb{C}[[x]]$ with $u(0) = 0$.

These solutions form a fundamental system unless

(a) b_0 has multiple roots and/or roots with integer difference, or

(b) $\deg b_0 < r$
(note: we always have $\deg b_0 \leq r$)

<u>Ad. (a)</u> Call two roots of b_0 equivalent if their difference is an integer. Then the set of roots of b_0 in $\overline{\mathbb{C}}$ splits into equivalence classes

$$\begin{array}{l} \nu_0, \nu_0 + k_{10}, \nu_0 + k_{20}, \ldots, \nu_0 + k_{\ell_0 0} \\ \nu_1, \nu_1 + k_{11}, \nu_1 + k_{21}, \ldots, \nu_1 + k_{\ell_1 1} \\ \vdots \end{array} \qquad \text{with } k_{ij} \in \mathbb{N}.$$

Let $\nu \in \bar{c}$ be the maximal root of one of the equivalence classes (so that $b_0(\nu) = 0$ and $b_0(\nu+k+1) \neq 0 \ \forall k \in \mathbb{N}$), and let m be its multiplicity.

Consider $u = \sum_{n=0}^{\infty} u_n x^n \in C(q)[[x]]$ defined via

$$b_0(n+q)u_n + \dots + b_{-m}(n+q)u_{n-m} = 0 \qquad (n>0)$$

$$u_0 = 1, \quad u_{-1} = u_{-2} = \dots = 0$$

Then for each $n \in \mathbb{N}$ we have

$$\operatorname{denom}(u_n) \mid b_0(q+1)\, b_0(q+2) \cdots b_0(n+q)$$

and for the generalized series object

$$y = x^q u(q,x)$$

we have

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = b_0(q)\, x^q.$$

Clear: setting $q = \nu$ turns y to a solution. Furthermore, since ν is an m-fold root of b_0, ~~and~~ $\left[\left(\frac{d}{dq}\right)^i b_0\right]_{q=\nu} = 0$ for $i = 0 \dots m-1$

Using the formal derivation rule

$$\frac{d}{dq} x^q = x^q \log x$$

it follows that

$$\left[\left(\frac{d}{dq}\right)^i (b_0 \cdot x^q) \right]_{q=\nu} = 0$$

for $i = 0 \dots m-1$.

Consequently, $\left[\left(\frac{d}{dq}\right)^i x^q u(q,x) \right]_{q=\nu}$ can be regarded as a formal solution of the DE for $i = 0 .. m-1$. (Note: the evaluation is legitimate because $q - \nu \nmid \mathrm{denom}(u_n)$ for all $n \in \mathbb{N}$.)

We thus obtain m solutions

$$y_1 = x^\nu u(\nu, x)$$

$$y_2 = x^\nu u_1(\nu, x) + x^\nu \log x \; u(\nu, x)$$

$$y_3 = x^\nu u_{1,1}(\nu, x) + 2x^\nu \log x \; u_1(\nu, x) + x^\nu (\log x)^2 u(\nu, x)$$

$$\vdots$$

$$y_m = \sum_{i=0}^{m-1} x^\nu (\log x)^i \binom{m-1}{i} \left[\left(\frac{d}{dq}\right)^i u(q,x) \right]_{q=\nu}.$$

A slightly more elaborate calculation shows that the ~~other~~ ~~so~~ non-maximal elements of an equivalence class lead to the same type of solutions.

(Details: Ince, Sect. 16.3)

Summarizing:

<u>Thm:</u> If $b_0 = (n-\nu_1)^{e_1} \cdots (n-\nu_\ell)^{e_\ell}$ is a factorization of ~~the~~ b_0 over $\mathbb{C}$, then there are generalized series solutions

$$x^{\nu_i} (\log x)^j u_{ij}(x)$$

$(i = 1..\ell,\ j = 0..e_i - 1)$ for $u_{ij} \in \mathbb{C}(\nu_i)[[x]]$.

<u>Remark:</u> b_0 is also called indicial equation of the DE, but not to be confused with the indicial equation introduced in II.1. (The latter appears in the present context as $b_{-m}(n+m) \in \mathbb{C}[n]$.)

$$\cdots \ast\ \ast\ 1\ 0\ 0\ 0\ 0\ 0$$