(4c) Polynomial Part

Task: Given $p \in k[x]$ (resp. $p \in k[x, x^{-1}]$ if x is exponential over k), find, if possible, $q \in k[x]$ (resp. $q \in k[x, x^{-1}]$) with $p - D(q) \in k$.

If no such q exists, then p is not elementary integrable.

If a q exists, it reduces the integration problem for $p \in k[x]$ (originally: $f \in K = k(x)$) to an integration problem for an element of k, which can be solved by applying the Risch algorithm recursively.

If x is exponential over k, say $Dx = ux$, then

$$D(q_m x^m) = D(q_m)x^m + m x^{m-1} q_m u x$$
$$= (D(q_m) + m u q_m) x^m$$

for every $m \in \mathbb{Z}$ and every $q_m \in k$, so in this case, we can consider each term.

of p separately: for each term $q_m x^m$ in p, we have to find $q_m \in k$ st

$$\boxed{D(q_m) + m u\, q_m = p_m}$$

or prove that no such q_m exists.

If x is logarithmic over k, say $Dx = u \in k \setminus C$, then $D(q_m x^m) = D(q_m)\, x^m + m\, x^{m-1} q_m u$.

A choice made for q_m (if there is a choice) at degree m influences the equations (and perhaps their solvability) for the lower degree terms.

<u>Ex:</u> $p = 1 + 2x + 3x^2 \qquad D(x) = u$

$q = q_0 + q_1 x + q_2 x^2 + q_3 x^3$

coefficient comparison:

x^3: $0 = D(q_3)$ $\longrightarrow$ any $q_3 \in C$ is OK here

x^2: $3 = D(q_2) + 3q_3 u$ $\rightarrow$ a solution $q_2 \in k$ of this equation may exist only for certain choices of $q_3 \in C$.

x^1: $2 = D(q_1) + 2q_2 u$

x^0: $1 = D(q_0) + q_1 u$

And if there is any solution $q_2 \in k$, then also $q_2 + c$ is a solution for every $c \in C$, providing a degree of freedom for the following equation.

In general: given $u, v \in k$, we need to decide whether there exist $w \in k$ and $c \in C$ st

$$\boxed{D(w) = u + cv}$$

More generally, completing the problems arising for exp and log extensions, we can solve the polynomial part problem if we can solve the following problem:

<u>Given:</u> $u \in k$, $f_1, \dots, f_m \in k$
<u>Find:</u> all $(g, c_1 \dots c_m) \in k \times C^m$ such that

$$D(g) + ug = c_1 f_1 + \dots + c_m f_m$$

<u>Note:</u>
(1) The solution set of this problem is a vector space over C. We seek a basis of this VS as output.

(2) There is no x any more in this problem, i.e. the problem is really one in k, not in $K = k(x)$.

Rename	$K := C(x_1 \dots x_{n-1})$ (= the former k)
	$x := x_{n-1}$
	$k := C(x_1 \dots x_{n-2})$

Then $u, g, f_1 \dots f_m$ belong to $K = k(x)$.
The problem is solved as follows (sketch):

(i) find $q \in k[x]$ such that any potential solution $(g, c_1 \dots c_m) \in K \times C^m$ is such that the denominator of g divides q.

(ii) Set $g = \frac{p}{q}$ with unknown $p \in k[x]$.
Then $D(g) = \frac{D(p)q - pD(q)}{q^2}$, so the problem becomes

$$\frac{1}{q} D(p) + \left(u - \frac{D(q)}{q^2}\right) p = c_1 f_1 + \dots + c_m f_m$$

After clearing denominators, we are left with

$$a\, D(p) + b\, p = c_1 h_1 + \dots + c_m h_m \qquad (*)$$

for known $a, b, h_1 \dots h_m \in k[x]$
and unknown $p \in k[x]$, $c_1 \dots c_m \in C$.

(iii) find $n \in \mathbb{N}$ such that any potential solution $(p, c_1 \dots c_m) \in k[x] \times C^m$ is such that $\deg p \leq n$.

(iv) • Write $p = p_n x^n + p_{rest}$ with undetermined $p_n \in k$ and $p_{rest} \in k[x]$ with $\deg(p_{rest}) \leq n-1$.

• Plug this ansatz into (*).
• Compare coefficients of the highest power of x appearing in the resulting equation. This yields an equation of the form

$$\alpha D(p_n) + \beta p_n = c_1 \eta_1 + \dots + c_m \eta_m$$

for known $\alpha, \beta, \eta_1 \dots \eta_m \in k$ and unknown $p_n \in k$, $c_1 \dots c_m \in C$.

• Solve this equation recursively. Let $\{(p_n^1, c_1^1 \dots c_m^1), \dots, (p_n^\ell, c_1^\ell \dots c_m^\ell)\}$ be a basis of the solution space.

- Setting

$$p_n := d_1 p_n^1 + \cdots + d_\ell p_n^\ell$$
$$c_1 := d_1 c_1^1 + \cdots + d_\ell c_1^\ell$$
$$\vdots$$
$$c_m := d_1 c_m^1 + \cdots + d_\ell c_m^\ell$$

into (*) yields an equation of the form

$$\square\, D(p_{rest}) + \square\, p_{rest} = d_1 \square + \cdots + d_\ell \square$$

with unknown $p_{rest} \in k[x]$ and $d_1, \ldots, d_\ell \in C$ and known $\square \in k[x]$.

- Determine the other coeffs of p_{rest} in the same way. ~~[illegible]~~ Each solution $(p_{rest}, d_1, \ldots, d_\ell) \in k[x] \times C^\ell$ corresponds to a solution

$$\begin{aligned}(p_{rest} &+ (d_1 p_n^1 + \cdots + d_\ell p_n^\ell) x^n, \\ &d_1 c_1^1 + \cdots + d_\ell c_1^\ell, \\ &\ldots \\ &d_1 c_m^1 + \cdots + d_\ell c_m^\ell) \in k[x] \times C^m\end{aligned}$$

of (*).

(v) If $\{(p^i, c_1^i, \dots, c_m^i) : i = 1..d\}$ is a basis of the solution space of (*), return $\{(\frac{p^i}{q}, c_1^i, \dots, c_m^i) : i = 1 \dots d\}$ as basis of the solution space of the original equation.

<u>Remark:</u>

Steps (i) and (iii) are nontrivial. We do not discuss them here, but we will later see how to do them in some simpler situations.

<u>Ex:</u>

Let $p = y \in \mathbb{Q}(x)[y]$ with $Dy = 2xy$
("$y = e^{x^2}$").

If p has an elementary integral, it must be of the form $q = q_1 y$ for some $q_1 \in \mathbb{Q}(x)$ (note that y is exponential over $\mathbb{Q}(x)$)

$$D(q) = D(q_1)y + q_1 2x y \overset{!}{=} y$$

Need: $q_1 = \frac{u}{v} \in \mathbb{Q}(x)$ st

$$\underbrace{\left(\frac{u}{v}\right)' + 2x\frac{u}{v}}_{= \frac{u'v - uv'}{v^2} + 2x\frac{u}{v}} = 1.$$

For such a pair $(u,v) \in \mathbb{Q}[x]$ we must have

$$u'v - uv' + 2xuv = v^2$$

$$(u' + 2xu - v)v = uv'$$

$$\Rightarrow v \mid uv' \underset{\gcd(u,v)=1}{\Longrightarrow} v \mid v' \text{ impossible for } v \in \mathbb{Q}[x] \setminus \mathbb{Q}.$$

$\Rightarrow v \in \mathbb{Q}$. wlog $v = 1$.

It remains to find $u \in \mathbb{Q}[x]$ with

$$u' + 2xu = 1.$$

But if $u \in \mathbb{Q}[x]$ has degree n, then $\deg(u') < n$ and $\deg(2xu) = n+1$, so $\deg(\text{lhs}) = n+1 \neq 0 = \deg(\text{rhs})$.

Therefore, no solution u can exist.

Therefore e^{x^2} is not elementary integrable.