III Operator Methods

1 "Closure Properties"

Let (K, D) be a differential field and $C = \mathrm{Const}\, K$. Until now we considered the problem of "finding" explicit solutions y of a given ODE

$$y^{(r)} + a_{r-1} y^{(r-1)} + \dots + a_0 y = 0$$

with coefficients in K.

Now: use the equation itself as an implicit description of its solution(s), either together with a suitable number of initial values of its Taylor expansion to separate it from the other solutions, or simply as a "generic" element of the solution space.

(Compare: implicit definition of an algebraic number by its minimal polynomial and, optionally, a sufficiently accurate approximation)

Let $K[\partial]$ be the set of univariate polynomials in the variable ∂ with coefficients in K.

Define addition on $K[\partial]$ as usual and multiplication via $\partial a = a\partial + D(a)$ $(a \in K)$ so that, for example,

$$(a_0 + a_1\partial + a_2\partial^2)\cdot(b_0 + b_1\partial)$$

$$= a_0b_0 + a_0b_1\partial + a_1\underbrace{\boxed{\partial b_0}}_{= b_0\partial + D(b_0)} + a_1\overbrace{\boxed{\partial b_1}}^{= b_1\partial + D(b_1)}\partial + a_2\underbrace{\boxed{\partial^2 b_0}}_{\substack{= \partial(b_0\partial + D(b_0)) \\ = b_0\partial^2 + 2D(b_0)\partial + D^2(b_0)}} + a_2\overbrace{\boxed{\partial^2 b_1}}^{= b_1\partial^2 + 2D(b_1)\partial + D^2(b_1)}\partial$$

$$= a_0b_0 + a_1D(b_0) + a_2D^2(b_0)$$
$$+ \left(a_0b_1 + a_1b_0 + a_1D(b_1) + 2a_2D(b_0) + a_2D^2(b_1)\right)\cdot\partial$$
$$+ \left(a_1b_1 + a_2b_0 + 2a_2D(b_1)\right)\cdot\partial^2$$
$$+ a_2b_1\partial^3$$

(Note: $lc(L_1L_2) = lc(L_1)\cdot lc(L_2)$ and $\deg(L_1L_2) = \deg(L_1) + \deg(L_2)$ for all $L_1, L_2 \in K[\partial]$ despite the noncommutativity)

Motivation: If F is a space of "functions" (more precisely, a K-VS with a function $\partial : F \to F$ satisfying $\partial(a \cdot f) = D(a) f + a\,\partial(f)$ for all $a \in K$, $f \in F$) and we let elements of $K[\partial]$ act on elements of F via

$$(a_0 + a_1 \partial + \dots + a_n \partial^n) \bullet f := a_0 f + a_0\, \partial(f) + \dots + a_n \partial^n(f)$$

then we have

$$\overbrace{(L_1 \underset{\substack{\uparrow \\ \text{mul in} \\ K[\partial]}}{\cdot} L_2)}^{\in K[\partial]} \underset{\substack{\uparrow \\ \text{appl}}}{\bullet} f = L_1 \underset{\substack{\uparrow \\ \text{appl}}}{\bullet} \overbrace{(L_2 \underset{\substack{\uparrow \\ \text{appl}}}{\bullet} f)}^{\in F}$$

for all $L_1, L_2 \in K[\partial]$ and $f \in F$.

Def:

(1) $f \in F$ is called a solution of $L \in K[\partial]$ $:\Leftrightarrow L \bullet f = 0$

(2) For $L \in K[\partial]$, $V(L) := \{ f \in F \mid L \bullet f = 0 \}$ is called the solution space of L

(3) For $f \in F$, $\mathrm{ann}(f) := \{L \in K[\partial] \mid L \bullet f = 0\}$ is called the annihilator of f.

(4) $f \in F$ is called D-finite
$:\Leftrightarrow \exists\, L \in K[\partial] \setminus \{0\} : \ L \bullet f = 0.$
($\Leftrightarrow$ The K-VS generated by $f, \partial f, \partial^2 f, \dots$ has finite dimension)

Note:

(1) If $L = L_1 L_2$ and f is a solution of L_2 then f is also a solution of L, because $L \bullet f = (L_1 L_2) \bullet f = L_1 \bullet (\underbrace{L_2 \bullet f}_{=0}) = 0$.

(2) If $L = L_1 + L_2$ and f is a common solution of L_1 and L_2, then f is also a solution of L, because $0 + 0 = 0$.

It follows that $\mathrm{ann}(f)$ is a left ideal in $K[\partial]$.

For any two $L_1, L_2 \in K[\partial]$ with $L_2 \neq 0$ we can find unique $Q, R \in K[\partial]$ with

$$L_1 = Q L_2 + R \quad \text{and} \quad \deg R < \deg L_2$$

by repeatedly subtracting suitable multiples $a \partial^i L_2$ from L_1, like in the commutative case. We write $\mathrm{lquo}(L_1, L_2) := Q$, $\mathrm{lrem}(L_1, L_2) := R$.

If I is a left ideal in $K[\partial]$ and $L_1, L_2 \in I$, then also $L_1 - Q L_2 = R \in I$. It follows that if G denotes a minimal order element of $I \setminus \{0\}$, then any other element L of I can be written $L = Q \cdot G$, because $L = QG + R$ with some $R \neq 0$ would yield a lower order element R of I in contradiction to the choice of G.

Furthermore, if G_1, G_2 are two monic least order elements of $I \setminus \{0\}$, then $G_1 = G_2$, for otherwise $G_1 - G_2$ would be a lower order element.

It follows that for every ideal $I \subseteq K[\partial]$ there exists a unique monic operator $G \in I$ with $I = \langle G \rangle$. For given $L_1, L_2 \in K[\partial]$, a $G \in K[\partial]$ with $\langle L_1, L_2 \rangle = \langle G \rangle$ can be computed by a straightforward adaption of the Euclidean algorithm. We call $G := \mathrm{gcrd}(L_1, L_2)$ the greatest common right divisor of L_1 and L_2.

<u>Observe:</u> If $G = \mathrm{gcrd}(L_1, L_2)$ then

$$V(G) = V(L_1) \cap V(L_2).$$

Conversely, if $L \in K[\partial]$ is the unique monic operator with $\langle L \rangle = \langle L_1 \rangle \cap \langle L_2 \rangle$, then L is called the least common left multiple of L_1 and L_2, $L := \mathrm{lclm}(L_1, L_2)$, and we have

$$V(L) = V(L_1) + V(L_2),$$

i.e. the solutions of L are linear combinations of solutions of L_1 ~~or~~ L_2.

Note: the formula $\operatorname{lcm}(p,q) = \frac{p\,q}{\gcd(p,q)}$ does not carry over to $K[\partial]$. But we can compute $\operatorname{lclm}(L_1, L_2)$ as follows:

- Make an ansatz $L = l_0 + l_1\partial + \dots + l_n\partial^n$ with undetermined coeffs l_i.
- Compute $L \operatorname{lrem} L_1 = \square + \square\partial + \dots + \square\partial^{\deg L_1 - 1}$
  $L \operatorname{lrem} L_2 = \square + \square\partial + \dots + \square\partial^{\deg L_2 - 1}$
- The $\square$ are certain K-linear combinations of the undetermined coefficients l_i. Forcing them to zero gives a linear system of equations over K with $n+1$ variables and $\deg L_1 + \deg L_2$ equations. Solve it.
- If there is no nontrivial solution, set $n = n+1$ and try again.

The smallest n for which a solution is found gives the lclm. At latest this happens when $n = \deg L_1 + \deg L_2$.

In particular: The sum of two D-finite functions is D-finite.

If $f_1 \in V(L_1)$, $f_2 \in V(L_2)$ then $\mathrm{lclm}(L_1, L_2) \in \mathrm{ann}(f_1 + f_2)$,

but it may not be the generator.

Product. Suppose now that (F, ∂) is even a differential ring and let $f_1, f_2 \in F$ be D-finite, say $L_1 \circ f_1 = 0$, $L_2 \circ f_2 = 0$ for some $L_1, L_2 \in K[\partial]$ with $d_1 = \deg L_1$, $d_2 = \deg L_2$.

The derivatives $f_1, \partial f_1, \partial^2 f_1, \dots$ generate the same K-VS as $f_1, \partial f_1, \dots, \partial^{d_1 - 1} f_1$. Likewise for f_2. Consequently, the K-VS generated by all the products $(\partial^i f_1)(\partial^j f_2)$ $(i, j \in \mathbb{N})$

$$\begin{matrix} f_1 f_2 & f_1 f_2' & f_1 f_2'' & \cdots \\ f_1' f_2 & f_1' f_2' & f_1' f_2'' & \cdots \\ f_1'' f_2 & f_1'' f_2' & f_1'' f_2'' & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is in fact generated by

$(\partial^i f_1)(\partial^j f_2)$ $(i = 0 \dots d_1 - 1;\ j = 0 \dots d_2 - 1)$, so its dimension is at most $d_1 d_2$. Now observe that

$$
\begin{aligned}
&f_1 f_2 \\
&\partial(f_1 f_2) = (\partial f_1) f_2 + f_1 (\partial f_2) \\
&\partial^2(f_1 f_2) = (\partial^2 f_1) f_2 + 2(\partial f_1)(\partial f_2) + f_1(\partial^2 f_2) \\
&\quad \vdots
\end{aligned}
$$

all belong to this vector space. Therefore, the first $d_1 d_2 + 1$ must be linearly dependent over K. The linear dependence corresponds to an operator

$$\square + \square \partial + \dots + \square \partial^{d_1 d_2}$$

which annihilates $f_1 f_2$.

In particular: The product of D-finite functions is D-finite.

An annihilating operator for ~~$f_1 f_2$~~ $f_1 f_2$ can be computed from annihilating operators L_1, L_2 of f_1, f_2 by following the steps of this argument.

The (unique) monic least order operator which can be found in this way is called the symmetric product of L_1 and L_2. It does not depend on the choice $f_1 \in V(L_1)$, $f_2 \in V(L_2)$ but only on L_1 and L_2.

If L is the symmetric product of L_1, L_2 then

~~$\forall f_1, f_2$~~

$\forall f_1 \in V(L_1)\ \forall f_2 \in V(L_2)$:

$$L \in \operatorname{ann}(f_1 f_2).$$

But L need not be the generator of these annihilators.