Alg 1. (Hermite Reduction)

INPUT: $f \in C(x)$

OUTPUT: $g, h \in C(x)$ such that $f = D(g) + h$ and h has a square free denominator.

1. Write $f = p + \dfrac{a}{d}$ for $p, a, d \in C[x]$ with $\deg(a) < \deg(d)$.
2. Let $d = d_1 d_2^2 \cdots d_m^m$ be the square free decomposition of d.
3. $u = d/d_m^m;\ \ v = d_m;\ \ g = \int p$.
4. While $m \geq 2$ do:
5. $$b := -\frac{a}{(m-1)v'u} \bmod v;$$ $$c := \frac{a + (m-1)buv' - b'uv}{v}$$
6. $$g = g + \frac{b}{v^{m-1}};\ \ a = c;$$ $$u = u/d_{m-1}^{m-1};\ \ v = v\, d_{m-1};$$ $$m = m - 1$$
7. Return g and $h := f - D(g)$.

Alg 2. (Rothstein Trager)

k a differential field, $k(x)$ a transcendental logarithmic or exponential extension of k such that $C := \mathrm{Const}(k(x)) = \mathrm{Const}(k)$.
INPUT: $f = \frac{p}{q} \in k(x)$ with $\deg(p) < \deg(p)$, q monic square free, x logarithmic over k or x exponential over k and $x \nmid q$.
OUTPUT: $\gamma_1, \ldots, \gamma_n \in \bar{C}$ and $u, v_1, \ldots, v_n \in k(\gamma_1, \ldots, \gamma_n)[x]$ such that

$$f = u + \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$$

or $\perp$ if no such data exists.

1. Compute $R := \mathrm{Res}_x(q, p - z\, D(q)) \in k[z]$.
2. Let $\gamma_1, \ldots, \gamma_n \in \bar{C}$ be the roots of those monic irreducible factors of R which belong to $C[z]$.
3. For each $i = 1, \ldots, n$, set $v_i := \gcd(q, p - \gamma_i\, D(q)) \in C(\gamma_i)[x]$.
4. $u := f - \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$.
5. If $u \in k[z]$, return $\gamma_1, \ldots, \gamma_n, u, v_1, \ldots, v_n$. Otherwise return $\perp$.

Alg 3. (Parameterized Risch Differential Equation; Sketch)

$k(x)$ a Liouvillean field, $C := \mathrm{Const}(k(x))$.
INPUT: $u, v, f_1, \ldots, f_m \in k(x)$.
OUTPUT: A basis of the C-vector space

$$\Big\{ \big(g, c_1, \ldots, c_m\big) \in k(x) \times C^m : u\,D(g) - v\,g = c_1\,f_1 + \cdots + c_m\,f_m \Big\}.$$

1. **(Denominator bound)** Find $q \in k[x]$ such that any potential solution $(g, c_1, \ldots, c_m) \in k(x) \times C^m$ is such that the denominator of g divides q.

2. Set $g = p/q$ with unknown $p \in k[x]$. Bring the equation into the form
$$a\,D(p) + b\,p = c_1\,h_1 + \cdots + c_m\,h_m \tag{1}$$
for known $a, b, h_1, \ldots, h_m \in k[x]$ and unknown $p \in k[x]$, $c_1, \ldots, c_m \in C$.

3. **(Degree bound)** Find $n \in \mathbb{N}$ such that any potential solution $(p, c_1, \ldots, c_m) \in k[x] \times C^m$ is such that $\deg(p) \leq n$.

4. Write $p = p_n x^n + p_{\text{rest}}$ with unknown $p_n \in k$ and $p_{\text{rest}} \in k[x]$ with $\deg(p_{\text{rest}}) \leq n-1$. Plug this ansatz into (1) and compare coefficients with respect to the highest power of x. This yields an equation of the form
$$\tilde{u}\,D(p_n) + \tilde{v}\,p_n = c_1\tilde{f}_1 + \cdots + c_m\tilde{f}_m$$
for known $\tilde{u}, \tilde{v}, \tilde{f}_1, \ldots, \tilde{f}_m \in k$ and unknown $p_n \in k$ and $c_1, \ldots, c_m \in C$.

5. Solve this equation recursively. Let
$$\Big\{ (p_n^1, c_1^1, \ldots, c_m^1),\ \ldots,\ (p_n^\ell, c_1^\ell, \ldots, c_m^\ell) \Big\}$$
be a basis of the solution space.

6. Set

$$\begin{aligned} p_n &= d_1\, p_n^1 + \cdots + d_\ell p_n^\ell \\ c_1 &= d_1\, c_1^1 + \cdots + d_\ell c_1^\ell \\ &\vdots \\ c_m &= d_1 c_m^1 + \cdots + d_\ell c_m^\ell \end{aligned}$$

with unknown $d_1, \dots, d_\ell \in C$. Plug this ansatz into (1) and obtain a new equation of the form

$$\bigcirc D(p_{\mathsf{rest}}) + \bigcirc p_{\mathsf{rest}} = d_1 \bigcirc + \cdots + d_\ell \bigcirc \qquad (2)$$

with unknown $p_{\mathsf{rest}} \in k[x]$ of degree at most $n-1$, unknown $d_1, \dots, d_\ell \in C$, and known $\bigcirc \in k[x]$.

7. Repeat steps 4–6 to determine all the coefficients of p_{rest}. Each solution $(p_{\mathsf{rest}}, d_1, \dots, d_\ell) \in k[x] \times C^\ell$ of (2) corresponds to a solution $(p_n x^n + p_{\mathsf{rest}}, c_1, \dots, c_m) \in k[x] \times C^m$ of (1), with $p_n, c_1, \dots, c_m$ as in 6.

8. Finally, if

$$\Big\{ (p^1, c_1^1, \dots, c_m^1),\ \dots,\ (p^d, c_1^d, \dots, c_m^d) \Big\}$$

is a basis of the solution space of (1), then

$$\Big\{ \Big(\frac{p^1}{q}, c_1^1, \dots, c_m^1\Big),\ \dots,\ \Big(\frac{p^d}{q}, c_1^d, \dots, c_m^d\Big) \Big\}$$

is a basis of the solution space of the original equation.

Alg 4. (Polynomial Solutions of fixed degree of linear ODEs)

INPUT: $a_0, \dots, a_r \in C[x]$, $a_r \neq 0$, $d \in \mathbb{N}$
OUTPUT: A basis of the C-vector space V_d consisting of all $p \in C[x]$ with $\deg(p) \leq d$ and $a_0 p + a_1 D(p) + \cdots + a_r D^r(p) = 0$, where $D = \frac{d}{dx}$.

1. Ansatz:
$$p = p_0 + p_0 x + \cdots + p_d x^d$$
with undetermined coefficients p_i.

2. Plug the ansatz into the ODE and collect coefficients with respect to x. Using that
$$D(y) = \sum_{i=0}^{d} p_i \, i x^{i-1},$$
$$D^2(y) = \sum_{i=0}^{d} p_i \, i(i-1) x^{i-2}, \text{ etc.}$$
regardless of the values of p_i, we obtain an equation of the form
$$\sum_{i=0}^{n} \left(\sum_{i=0}^{d} c_{i,j} p_i \right) x^j = 0$$
for some explicit $c_{i,j} \in C$.

3. Equate coefficients of x^j to zero, i.e., compute the nullspace of
$$A := \begin{pmatrix} c_{0,0} & \cdots & c_{d,0} \\ \vdots & \ddots & \vdots \\ c_{0,n} & \cdots & c_{d,n} \end{pmatrix} \in C^{n \times d}$$
We have
$$(p_0, \dots, p_d) \in \ker A \quad \Longleftrightarrow \quad p_0 + p_1 x + \cdots + p_d x^d \in V_d$$

Alg 5. (Rational solutions of linear ODEs)

INPUT: $a_0, \ldots, a_r \in C[x]$, $a_r \neq 0$.
OUTPUT: A basis of the C-vector space V consisting of all $y \in C(x)$ such that $a_0 y + a_1 D(y) + \cdots + a_r D^r(y) = 0$, where $D = \frac{d}{dx}$.

1. Let $q_1, \ldots, q_m \in C[x]$ be the monic irreducible factors of a_r.
2. For $i = 1, \ldots, m$, let e_i be the smallest integer root of the indicial polynomial of the ODE with respect to q_i (or $e_i = 0$ if there are no integer roots)
3. Make an ansatz $y = q_1^{e_1} q_2^{e_2} \cdots q_m^{e_m}\, p$ for an unknown polynomial p, plug it into the ODE, clear denominators, and obtain a linear ODE for p.
4. Find a basis $B := \{p_1, \ldots, p_d\} \subseteq C[x]$ of the C-vector space of all polynomial solutions of this new equation.
5. Return $\{q_1^{e_1} q_2^{e_2} \cdots q_m^{e_m}\, p : p \in B\} \subseteq C(x)$

Alg 6. (Generalized series solutions of linear ODEs)

INPUT: $a_0, \dots, a_r \in C[x^{1/q}]$, $a_r \neq 0$, a bound B (initially $B = -\infty$).
OUTPUT: A fundamental system of the ODE $a_0 y + a_1 D(y) + \cdots + a_r D^r(y) = 0$, where $D = \frac{d}{dx}$, where each element is of the form

$$\exp(c_1 x^{k_1} + \cdots + c_\ell x^{k_\ell}) x^\nu u(x, \log x)$$

with $B < k_1 < \cdots < k_\ell < 0$, $c_1, \dots, c_\ell \in \bar{C} \setminus \{0\}$, $\nu \in \bar{C}$, $u \in \bar{C}[[x]][z]$ (truncated).

1. $S :=$ a basis of the space of solutions of the form $x^\nu u(x, \log x)$.
2. Using the Newton polygon, determine the eligible values k.
3. For all eligible values k with $B < k$ do:
4. Find the corresponding values $c \in \bar{C} \setminus \{0\}$
5. For each such c do:
6. substitute $y \leftarrow e^{cx^k} \tilde{y}$ into the equation and divide by e^{cx^k} to obtain an new linear ODE with (Puiseux-)polynomial coefficients and $\tilde{y}$ as unknown function
7. call the algorithm recursively for this ODE and $B = k$. Let $\tilde{S}$ be the output.
8. $S = S \cup \{\mathrm{e}^{cx^k} \tilde{y} : \tilde{y} \in \tilde{S}\}$
9. return S.

Alg 7. (Hyperexponential solutions of linear ODEs)

INPUT: $a_0, \dots, a_r \in C[x]$.
OUTPUT: All hyperexponential solutions y of the ODE $a_0 y + a_1 D(y) + \cdots + a_r D^r(y) = 0$, where $D = \frac{d}{dx}$.

1. $E := \{1\}$.
2. Let $c_0 = \infty$ and let $c_1, \dots, c_\ell \in \bar{C}$ be the roots of a_r.
3. For $i = 0, \dots, \ell$ do:
4. apply a change of variables $x' = 1/x$ (if $i = 0$) or $x' = x - c_i$ (if $i > 0$) to the input ODE.
5. use (parts of) Alg. 6 to compute the set P of all $p \in \bar{C}[x']$ such that the new ODE has a solution with exponential part $\mathrm{e}^{p(1/x')}$.
6. $E = \{a\mathrm{e}^{p(1/x')} : a \in E, p \in P\}$.
7. $S = \emptyset$
8. For each $\mathrm{e}^v \in E$ do:
9. apply a substitution $y = \mathrm{e}^v \tilde{y}$ to the input ODE.
10. use (a slight variation of) Alg. 5 to compute the set R of all solutions $p_1^{e_1} \cdots p_n^{e_n}$ of the new ODE, with $p_1, \dots, p_n \in \bar{C}[x]$ monic and $e_1, \dots, e_n \in \bar{C}$ (sic!)
11. $S := S \cup \{\mathrm{e}^v p : p \in R\}$
12. return S

Alg 8. (Least Common Left Multiple)

INPUT: $L_1, L_2 \in K[\partial]$
OUTPUT: the least common left multiple of L_1 and L_2.

1. If $\deg_\partial(L_1) > \deg_\partial(L_2)$ then exchange L_1 and L_2 (optional)
2. $R = [\mathrm{rem}(L_2, L_1)]$; $n := 0$
3. while R is linearly independent over K do:
4. $\quad n := n + 1$
5. $\quad$ append $\mathrm{rem}(\partial^n L_1, L_2)$ to R
6. Let $a_0, \ldots a_n$ not all zero such that $\sum_{k=0}^{n} a_k R[k] = 0$.
7. Return $(a_0 + a_1\partial + \cdots + a_n\partial^n)L_1$

Alg 9. (Symmetric Product)

INPUT: $L_1, L_2 \in K[\partial]$
OUTPUT: the symmetric product of L_1 and L_2.

1. $R = [1 \otimes 1]$; $n := 0$
2. while R is linearly independent over K do:
3. $\quad n := n + 1$
4. $\quad$ append $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \mathrm{rem}(\partial^k, L_1) \otimes \mathrm{rem}(\partial^{n-k}, L_2)$ to R
5. Let $a_0, \ldots, a_n$ not all zero such that $\sum_{k=0}^{n} a_k R[k] = 0$.
6. Return $a_0 + a_1\partial + \cdots + a_n\partial^n$

Alg 10. (D-finite addition via FGLM)

INPUT: 0-dimensional left ideals $\mathfrak{a}, \mathfrak{b} \subseteq K[\partial_1, \ldots, \partial_n]$
OUTPUT: $\mathfrak{a} \cap \mathfrak{b}$

1. $B = \emptyset$; $G = \emptyset$
2. Let $\tau = \partial_1^{e_1} \cdots \partial_n^{e_n}$ be the smallest term which is neither in B nor a multiple of $\operatorname{lt}(g)$ for some $g \in G$
3. If no such term exists, then G is a (Gröbner) basis of $\mathfrak{a} \cap \mathfrak{b}$.
   In this case, return G and stop.
4. If there exist elements $c_\sigma \in K$ with

   $$\operatorname{red}(\tau, \mathfrak{a}) = \sum_{\sigma \in B} c_\sigma \operatorname{red}(\sigma, \mathfrak{a}) \quad \text{and} \quad \operatorname{red}(\tau, \mathfrak{b}) = \sum_{\sigma \in B} c_\sigma \operatorname{red}(\sigma, \mathfrak{b})$$

   then add $\tau - \sum_{\sigma \in B} c_\sigma \sigma$ to G
   else add τ to B
5. go to 2.