③ Liouville's Theorem

<u>Goal:</u> Understand the possible elementary extensions E that may be needed to integrate some $f \in K$ when K is a Liouvillean field.

<u>Recall:</u> If $K = \mathbb{C}(x)$ and $D = \frac{d}{dx}$ then there is always an elementary integral of the form

$$g + \sum_{i=1}^{n} \gamma_i \log v_i$$

for some $g \in \mathbb{C}(x)$, $\gamma_i \in \overline{\mathbb{C}}$, $v_i \in \overline{\mathbb{C}}[x]$).

Liouville's theorem says essentially that an $f \in K$ either has an elementary integral of the form

$$g + \sum_{i=1}^{n} \gamma_i \log v_i$$

for some $g \in K$, $\gamma_i \in \mathrm{Const}\, K$, $v_i \in K$ (assuming $\mathrm{Const}\, K = \overline{\mathrm{Const}\, K}$ here for simplicity) or no elementary integral at all.

Recall: K is called a Liouvillean field if $K = C(x_1, \ldots, x_n)$ where each x_i is Liouvillean over $C(x_1, \ldots, x_{i-1})$ and $\operatorname{Const} K = C$.

Note: The requirement $\operatorname{Const} K = C$ is a nontrivial condition, because a naive construction may induce unwanted "fake constants".

Ex:

(1) $K = \mathbb{C}(x_1, \ldots x_4)$ with $x_1 \ldots x_4$ transcendental (i.e. algebraically independent) over $\mathbb{C}$.

$$D(c) = c \qquad (c \in \mathbb{C})$$
$$D(x_1) = 1 \qquad (\text{"} x_1 = t \text{"})$$
$$D(x_2) = \frac{1}{x_1 + 1} \qquad (\text{"} x_2 = \log(t+1) \text{"})$$
$$D(x_3) = \frac{1}{x_1 - 1} \qquad (\text{"} x_3 = \log(t-1) \text{"})$$
$$D(x_4) = \frac{2x_1}{x_1^2 + 1} \qquad (\text{"} x_4 = \log(t^2 - 1) \text{"})$$

Clearly $\mathbb{C} \subseteq \operatorname{Const} K$ by construction. But the inclusion is proper!

Consider

$$f = x_2 + x_3 - x_4$$

Then

$$D(f) = \frac{1}{x_1+1} + \frac{1}{x_1-1} - \frac{2x_1}{x_1^2-1} = 0,$$

so $f \in \mathrm{Const}\,K$ although $f \notin \mathbb{C}$.

(2) $K = \mathbb{C}(x_1, x_2, x_3)$ with x_1, x_2, x_3 alg. indep over $\mathbb{C}$.

$$D(c) = 0 \qquad (c \in \mathbb{C})$$

$$D(x_1) = 1 \qquad (\text{"} x_1 = t \text{"})$$

$$D(x_2) = \frac{1}{x_1} \qquad (\text{"} x_2 = \log t \text{"})$$

$$D(x_3) = \frac{1}{2}\frac{1}{x_1}x_3 \qquad (\text{"} x_3 = e^{\frac{1}{2}\log t} = \sqrt{t} \text{"})$$

Then $f = \frac{x_3^2}{x_1} \in \mathrm{Const}\,K \setminus \mathbb{C}$, because

$$D(f) = \frac{2x_3\,D(x_3)\,x_1 - x_3^2\,1}{x_1^2}$$

$$= \frac{2x_3^2\,\frac{1}{2}\,\frac{1}{x_1}\,x_1 - x_3^2}{x_1^2} = 0.$$

However, $g = x_3^2 - x_1 \notin \mathrm{Const}\,K$:

$$D(g) = 2x_3^2\frac{1}{2x_1} - 1 = \frac{x_3^2 - x_1}{x_1} \neq 0.$$

If such strange constants appear, it means that the field K was not constructed in the right way. For Liouvillean fields, it is assumed by Definition that this situation does not happen.

Main idea behind Liouville's theorem: if $f \in K$ and $g \in E$ is such that $f = D(g)$ then all parts of g which do not belong to K must disappear in the differentiation. Most extensions do not have this feature.

<u>Lemma 1</u> Let K be a differential field and $E = K(x)$ be a hyperexponential transcendental extension of K with $\mathrm{const}\, K = \mathrm{const}\, E$, say $D(x) = u \cdot x$ for some $u \in K \setminus \{0\}$. Let $g \in E \setminus K$.

Then $D(g) \in E \setminus K$.

Proof. First assume that $g \in K[x]$ with $\deg g > 0$, say

$$g = g_0 + g_1 x + \cdots + g_d x^d$$

for some $d > 0$ and $g_d \neq 0$. Then

$$D(g) = \delta_D(g) + \frac{d}{dx}(g) \cdot u \cdot x.$$

If $D(g) \in K$, then the coefficient of x^d in the latter expression must be zero:

$$D(g_d) - du\, g_d = 0$$
$$D(g_d) = du\, g_d.$$

But then

$$D\left(\frac{g_d}{x^d}\right) = \frac{D(g_d)x^d - g_d\, d\, x^{d-1} D(x)}{x^{2d}}$$
$$= \frac{du\, g_d x^d - du\, g_d x^d}{x^{2d}} = 0,$$

so $0 \neq \frac{g_d}{x^d} \in \operatorname{Const} E = \operatorname{Const} K \subseteq K$, which is in conflict with the transcendence of x.

If $g \in K(x) \setminus K[x]$, then we can write

$$g = p + \frac{a}{b}$$

for some $p, a, b \in K[x]$ with $a \neq 0$, $\deg b < \deg a$.
and $\gcd(a,b) = 1$

Then

$$D(g) = D(p) + \frac{D(a)b - aD(b)}{b^2}$$

This can only belong to K if $D(a)b - aD(b) = 0$, because $\deg(D(a)b - aD(b)) \leq \deg a + \deg b < 2\deg b$.

But then $\frac{a}{b} \in \mathrm{Const}\, E = \mathrm{Const}\, K \subseteq K$. This is not possible. $\boxtimes$

<u>Lemma 2</u> Let K be a differential field, $E = K(x)$ with x primitive transcendental over K, say $D(x) = u \in K$, and $\mathrm{Const}\, E = \mathrm{Const}\, K$.

Let $g \in E$ be such that $D(g) \in K$.

Then $g = cx + v$ for some $c \in \mathrm{Const}\, E$ and $v \in K$.

<u>Proof:</u> Let $g \in K[x]$ with $n = \deg g > 0$, say

$$g = g_0 + g_1 x + g_2 x^2 + \cdots + g_n x^n.$$

Then
$$D(g) = D(g_n) x^n + (D(g_{n-1}) + n g_n u) x^{n-1} + (D(g_{n-2}) + (n-1) g_{n-1} u) x^{n-2} + \cdots$$

If $n > 0$ and $g_n \notin \mathrm{Const}\, K$, then $D(g) \notin K$.
If $n > 1$ and $g_n \in \mathrm{Const}\, K$, then for $D(g) \in K$

we would need
$$D(g_{n-1}) + n g_n u = 0$$

But then
$$D(\underbrace{g_{n-1} + n g_n x}_{\in E \setminus K}) = 0$$

in contradiction to $\mathrm{Const}\, K = \mathrm{Const}\, E$.

Therefore $D(g)$ is only possible if $n \leq 1$ and $g_1 \in \mathrm{Const}\, K$.

For $g = p + \frac{a}{b} \in K(x) \setminus K[x]$, use the same argument as in Lemma 1. ▣

<u>Lemma 3</u> Let K be a differential field and $E = K(x)$ with x algebraic over K. Let $f \in K$. If there exists $g \in E$ with $D(g) = f$ then there also exists $g \in K$ with $D(g) = f$.

<u>Proof.</u> First note that the action of D on x is uniquely determined by the action of D on K. For, if $P \in K[X]$ is the minimal polynomial of x, i.e. $P(x) = 0$, then, as $0 \in \mathrm{Const} K$, $D(P(x)) = 0$ too, so

$$0 = D(P(x)) = \delta_D(P) + \underbrace{\frac{d}{dx}(P)}_{\neq 0} \cdot D(x)$$

so $D(x) = -\dfrac{\delta_D(P)(x)}{\frac{d}{dx}(P)(x)}$. Therefore, if $\bar{x} \in \overline{K}$ is some conjugate of x, then necessarily $D(\bar{x}) = -\dfrac{\delta_D(P)(\bar{x})}{\frac{d}{dx}(P)(\bar{x})}$. It follows that when $g \in E$ is such that $D(g) = f \in K$, then

$$\begin{array}{ccc} \mathrm{Tr}(f) & = & \mathrm{Tr}(D(g)) \\ \| & & \| \\ f & \cdot & D(\underbrace{\mathrm{Tr}(g)}_{\in K}) \end{array}.$$

$\boxtimes$

in full generality:

Thm 5. (Liouville) Let K be a differential field with algebraically closed constant field C, and let $f \in K$. Then f is elementary integrable if and only if

$$f = D(g) + \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$$

for some $g \in K$, $\gamma_1 \ldots \gamma_n \in C$, $v_1 \ldots v_n \in K$.