② Several Variables

Let K be a field and $D_1, \dots, D_n : K \to K$ several derivations on K with $D_i \circ D_j = D_j \circ D_i$ ($i,j = 1..n$). The elements of K can then be interpreted as functions in several variables

Ex: $K = \mathbb{Q}(x_1 .. x_n)$ with $\frac{d}{dx_1}, \frac{d}{dx_2}, \dots, \frac{d}{dx_n}$.

Let $K[\partial_1, \dots, \partial_n]$ be the set of multivariate polynomials in $\partial_1 .. \partial_n$ with coefficients in K. Define addition on $K[\partial_1, .. \partial_n]$ as usual and multiplication via $\partial_i a = a\partial_i + D_i(a)$ ($a \in K$), $\partial_i \partial_j = \partial_j \partial_i$ ($i,j = 1..n$).

Let F be a K-VS with n maps $\partial_i : F \to F$ satisfying $\partial_i(af) = D_i(a)f + a\,\partial_i(f)$ for all $a \in K$, $f \in F$, $i = 1..n$.

Let $K[\partial_1 ... \partial_n]$ act on F in the natural way.

Def:

(1) $\mathrm{ann}(f) := \{ L \in K[\partial_1 .. \partial_n] \mid L \bullet f = 0 \}$ is called the annihilator of $f \in F$.

(2) A left ideal $\mathfrak{a}$ of $K[\partial_1 .. \partial_n]$ is called 0-dimensional iff $\forall i \in \{1..n\}: \mathfrak{a} \cap K[\partial_i] \neq \{0\}$.

(3) $f \in F$ is called D-finite
$:\Leftrightarrow \forall i \in \{1..n\} \; \exists L \in K[\partial_i] \setminus \{0\} : \; L \bullet f = 0$
($\Leftrightarrow$ $\mathrm{ann}(f)$ is 0-dimensional)

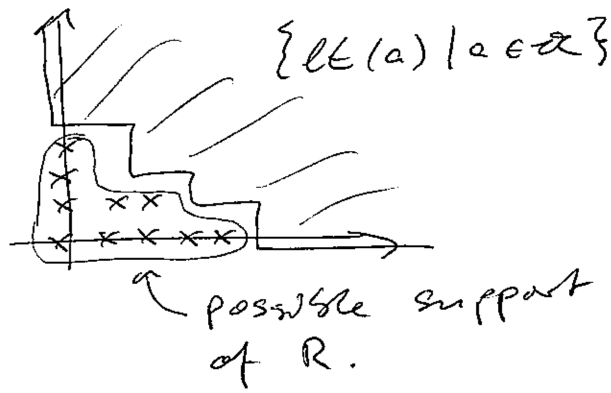
Ex: $e^{x \cdot y}$ is D-finite because

$$\underbrace{(\partial_x - y)}_{\in \mathbb{Q}(x,y)[\partial_x]} \bullet e^{xy} = 0 \quad \text{and} \quad \underbrace{(\partial_y - x)}_{\in \mathbb{Q}(x,y)[\partial_y]} \bullet e^{xy} = 0$$

As in the univariate case, $\mathrm{ann}(f)$ is a left ideal. However, $K[\partial_1 .. \partial_n]$ is no longer a PID if $n > 1$. Still, Gröbner bases theory provides us with an ordering of the power products $\partial_1^{e_1} \dots \partial_n^{e_n}$ extending the notion of leading terms, as well as an algorithm which for given $L \in K[\partial_1 .. \partial_n]$ and $\mathfrak{a} \subseteq K[\partial_1 .. \partial_n]$ computes $A, R \in K[\partial_1 .. \partial_n]$ with

$$L = A + R, \quad A \in \mathfrak{a}, \quad R = 0 \text{ or } lt(R) < lt(a) \text{ for all } a \in \mathfrak{a}.$$

Def: $red(L, \mathfrak{a}) := R$.



Closure properties:
If f, g are D-finite then so are $f+g$, $f \cdot g$.

Proof: apply the univariate argument for each $i = 1..n$ separately.

Computation: either as in the proof, or, more efficiently, by an FGLM-like algorithm, as follows. (for addition, multiplication works analogously):

Input: $\mathfrak{a}, \mathfrak{b} \subseteq K[\partial_1 \ldots \partial_n]$ 0-dim left ideals

Output: $\mathfrak{a} \cap \mathfrak{b}$

(1) $B = \emptyset$; $G = \emptyset$

(2) Let $\tau = \partial_1^{e_1} \cdots \partial_n^{e_n}$ be the smallest term which is neither in B nor a multiple of $lt(g)$ for some $g \in G$.

(3) If no such term exists, then G is a (Gröbner) basis of G. Return G and stop.

(4) If there exist elements c_σ of K with

$$\operatorname{red}(\tau, \mathfrak{a}) = \sum_{\sigma \in B} c_\sigma \operatorname{red}(\sigma, \mathfrak{a})$$

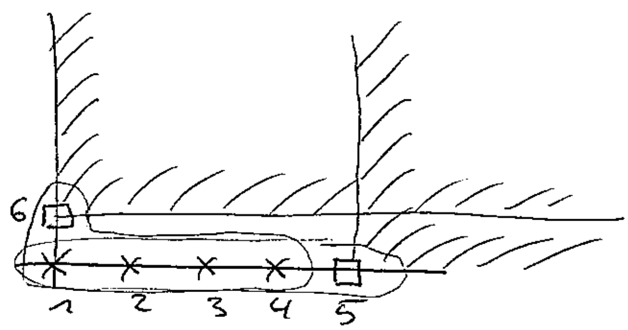
and $\operatorname{red}(\tau, \mathfrak{b}) = \sum_{\sigma \in B} c_\sigma \operatorname{red}(\sigma, \mathfrak{b})$

then add $\tau - \sum_{\sigma \in B} c_\sigma \cdot \sigma$ to G

else add τ to B

(5) goto (2).

Ex:



③ Definite Integration

Suppose $f(x,y)$ is a "function" and we ~~want~~ want to "compute" the integral

$$F(x) := \int_a^b f(x,y)\,dy .$$

This is easy if $f(x,y)$ is indefinitely integrable wrt y, i.e. $f(x,y) = D_y\, g(x,y)$ for some $g(x,y)$, because then $F(x) = g(x,b) - g(x,a)$. Suppose this is not the case.

Idea: Find some operator

$$L = c_0(x) + c_1(x)\partial_x + \dots + c_r(x)\partial_x^r$$

such that $L \circ f$ is integrable, say

$$c_0(x) f(x,y) + \dots + c_r(x)\, D_x^r f(x,y) = D_y\, g(x,y).$$

Then integrating both sides gives

$$c_0(x)\, F(x) + \dots + c_r(x)\, D_x^r F(x) = g(x,b) - g(x,a).$$

The latter is an inhomogeneous ODE for $F(x)$ from which further information (e.g. closed forms) can be extracted by other algorithms.

Task: given $\mathfrak{a} \subseteq K[\partial_x, \partial_y]$, find $L \in \mathrm{Const}_{D_y}(K)[\partial_x] \setminus \{0\}$ and $Q \in K[\partial_x, \partial_y]$ such that $L - \partial_y Q \in \mathfrak{a}$.

(imagine $\mathfrak{a} = \mathrm{ann}(f)$, $g = Qf$)

How? Consider for simplicity the case $f = \frac{p}{q} \exp(\frac{u}{v})$ for some $p, q, u, v \in C[x,y]$ with $\frac{u}{v} \notin C(x)$, $\deg_y p, q, u, v \leq n$.

Wlog $q = w^a v^b$ for $w \in C[x,y]$, $a, b \in \mathbb{N}$ we have

$$D_x(f) = \left(\frac{p'wv - aw'vp - bwv'p}{w^{a+1}v^{b+1}} + \frac{p}{w^a v^b} \frac{u'v - uv'}{v^2} \right) \exp\left(\frac{u}{v}\right)$$

$$= \frac{\mathrm{poly}}{w^{a+1}v^{b+2}} \exp\left(\frac{u}{v}\right)$$

for some poly of degree $\leq 3n$.

By induction, it follows that

$$D_x^k(f) = \frac{\text{poly}}{w^{a+k} v^{b+2k}} \exp\left(\frac{u}{v}\right)$$

for some poly of degree $\leq (2k+1)n$.

For $L \in C(x)[\partial_x]$ of order $r>0$, we therefore have

$$L \cdot f = \frac{\text{poly}}{w^{a+r} v^{b+2r}} \exp\left(\frac{u}{v}\right)$$

for some poly of y-degree $\leq (3r+1)n$.

If we set $Q = \frac{q_0 + q_1 y + \dots + q_d y^d}{w^{r-1} v^{2(r-1)}} f$, we have

$$\partial_y Qf = \frac{\text{poly}}{w^{a+r} v^{b+2r}} \exp\left(\frac{u}{v}\right)$$

for some poly of degree $d+3n$. Now make an ansatz

$$L = c_0 + c_1 \partial_x + \dots + c_r \partial_x^r$$

$$Q = \frac{q_0 + q_1 y + \dots + q_d y^d}{w^{r-1} v^{2(r-1)}}$$

with unknown $c_i, q_j \in C(x)$.

If we choose $d = (3r-2)n$, then

$$(L - \partial_y Q)\cdot f = \frac{\text{poly}}{w^{a+r}\, v^{b+2r}} \exp\left(\frac{u}{v}\right)$$

for some poly of y-degree $\le (3r+1)n$ whose coefficients are $C(x)$-linear combinations of the undetermined coefficients c_i, q_j. Forcing them to zero yields a linear system of equations over $C(x)$ with

$$(3r+1)n+1 \quad \text{equations}$$

$$\underbrace{(r+1)}_{\text{from } L} + \underbrace{(3r-2)n+1}_{\text{from } Q} \quad \text{variables}$$

which will have a solution as soon as

$$(3r-2)n + r + 2 > (3r+1)n+1$$

$$\iff \boxed{r > 3n-1}$$

Note: We can't have $L=0$ for this solution because then $Q\neq 0$ and $D_y(Qf)=0$ which is in conflict with the assumption $\frac{u}{v}\notin C(x)$ ($\Leftrightarrow$ f is not rational "wrt y", so Qf cannot be a constant wrt y)

We have thus shown that $L\neq 0$ and Q exist for every f of the form $\frac{p}{q}\exp(\frac{u}{v})$

More generally, the same reasoning can be used to show that $L\neq 0$ and Q exist and can be computed whenever $\mathrm{ann}(f)$ is 0-dim, viz. when f is D-finite.