Part 1

1 Basic Theory of Categories

1.1 Concrete Categories

One reason for inventing the concept of category was to formalize the notion of a set with additional structure, as it appears in several branches of mathematics. The structured sets are abstractly considered as objects. The structure preserving maps between objects are called morphisms.

Formulating all items in terms of objects and morphisms without reference to elements, many concepts, although of disparate flavour in different mathematical theories, can be expressed in a universal language. The statements about 'universal concepts' provide theorems in each particular theory.

1.1 Definition (Concrete Category)**.**
A concrete category is a triple $C = (\mathcal{O}, U, \hom)$*, where*

- $\mathcal{O}$ *is a class;*
- $U : \mathcal{O} \longrightarrow \mathbf{V}$ *is a set-valued function;*
- $\hom : \mathcal{O} \times \mathcal{O} \longrightarrow \mathbf{V}$ *is a set-valued function;*

satisfying the following axioms:

1. $\hom(a, b) \subseteq U(b)^{U(a)}$;
2. $\mathrm{id}_{U(a)} \in \hom(a, a)$;
3. $f \in \hom(a, b) \wedge g \in \hom(b, c) \Rightarrow g \circ f \in \hom(a, c)$.

The members of $\mathcal{O}$ are the **objects** of C. An element $f \in \hom(a, b)$ is called a **morphism** with domain a and codomain b. A morphism in $\hom(a, a)$ is an **endomorphism**. Thus

$$\mathrm{End\,a} = \hom(a, a).$$

We call $U(a)$ the underlying set of the object $a \in \mathcal{O}$. By Axiom 1, a morphism $f \in \hom(a, b)$ is a mapping $U(a) \longrightarrow U(b)$. We write $f\colon a \longrightarrow b$ or $a \xrightarrow{f} b$ to indicate that $f \in \hom(a, b)$. Axiom 2 ensures that every object a has the identity function $\mathrm{id}_{U(a)}$ as an endomorphism. Axiom 3 guaranties that morphisms can be composed.

1.2 Example (Concrete Categories)**.** *The following are examples of concrete categories.*

1. *$\mathcal{SET}$ is the category of sets. Its class of objects is $\mathbf{V}$. The mapping $U\colon \mathbf{V} \longrightarrow \mathbf{V}$ is the identity function $\mathbf{ID}|\mathbf{V}$. For arbitrary sets x, y we have* $\hom(x, y) = y^x$.
2. *$\mathcal{GROUP}$ is the category of groups. For any group g, $U(g)$ is the underlying set of g, that is, g with all operations ignored. If g, h are groups, then the set* $\hom(g, h)$ *is the set of all group homomorphisms $g \longrightarrow h$.*
3. *$\mathcal{RING}$ is the category of rings.*[1] *If Λ is a ring, then $U(\Lambda)$ is the underlying set of Λ. For rings Λ, Σ, the set* $\hom(\Lambda, \Sigma)$ *is the set of all ring homomorphisms $\Lambda \longrightarrow \Sigma$.*
4. *${}_\Lambda\mathcal{MOD}$, where Λ is a ring. It has all left unitary Λ-modules as objects. $U(a)$ is the underlying set of the module a; $hom(a, b)$ is the set of all Λ-linear homomorphisms.*[2]
5. *The category $\mathcal{TOP}$ whose object class consists of all topological spaces x, $U(x)$ is the set of all points of x; $hom(x, y)$ is the set of continuous functions $x \longrightarrow y$. Often we will write $C(x, y)$ for the set of continuous mappings $x \longrightarrow y$.*

In a similar way one obtains

$\mathcal{MOD}_\Lambda$	*right Λ-modules and Λ-homomorphisms;*
${}_\Lambda\mathcal{MOD}_\Sigma$	*$\Lambda\Sigma$-bimodules and $\Lambda\Sigma$-homomorphisms;*
$\mathcal{SGRP}$	*semigroups and their homomorphisms;*
$\mathcal{CSGRP}$	*commutative semigroups and semigroup homomorphisms;*
$\mathcal{MON}$	*monoids and identity-preserving semigroup homomorphisms;*
$\mathcal{CMON}$	*abelian monoids and monoid homomorphisms;*
$\mathcal{AB}$	*abelian groups and group homomorphisms;*
$\mathcal{CRING}$	*commutative rings and ring homomorphisms;*
$\mathcal{ID}$	*integral domains and ring homomorphisms;*
$\mathcal{FIELD}$	*fields and homomorphisms;*
$\mathcal{ALG}_R$	*associative R-algebras and algebra homomorphisms (where $R \in \mathcal{CRING}$);*
$\mathcal{CALG}_R$	*commutative R-algebras and algebra homomorphisms;*
$\mathcal{LIEALG}_k$	*Lie algebras and Lie homomorphisms ($k \in \mathcal{FIELD}$);*
$\mathcal{R}el$	*sets with a binary relation and relation preserving maps;*
$\mathcal{POS}$	*partially ordered sets and monotone functions;*
$\mathcal{LAT}_\sqcup$	*join semi-lattices and supremum preserving functions;*
$\mathcal{LAT}_\sqcap$	*meet semi-lattices and infimum preserving functions;*
$\mathcal{LAT}$	*lattices and lattice homomorphisms;*[3]
$\mathcal{BA}$	*Boolean algebras and Boolean homomorphisms;*
$\mathcal{BR}$	*Boolean rings and ring homomorphisms;*[4]
$\mathcal{BS}$	*Boolean spaces and continuous functions;*[5]

[1]'Ring' always means 'ring with unit', and a homomorphism of rings preserves units.

[2]If Λ is a field, this gives the category of vector spaces over Λ and Λ-linear maps.

[3]A lattice homomorphism is a map obeying the rule $f(x \sqcup y) = fx \sqcup fy$ and $f(x \sqcap y) = fx \sqcap fy$.

[4]A Boolean ring is a ring R with 1 and idempotent multiplication. Evidently R is commutative and obeys the rule $x + x = 0$.

[5]A Boolean space is a totally disconnected, compact Hausdorff space, where totally disconnected means that every point is its own connected component. Equivalently, a space is Boolean, iff it is 0-dimensional and compact and Hausdorff. Such a space is also called a **Stone space**.

$\mathcal{TOP}_{T0}$	*T0-spaces and continuous functions;*
$\mathcal{TOP}_{T1}$	*T1-spaces and continuous functions;*
$\mathcal{TOP}_{T2}$	*Hausdorff spaces and continuous functions;*
$\mathcal{TOP}_{Comp.T2}$	*compact Haussdorff spaces and continuous functions;*
$\mathcal{TOP}_{Loc.Comp}$	*locally compact Hausdorff spaces and continuous functions;*
$\mathcal{TOP}_{C.Reg}$	*completely regular spaces and continuous functions;*[6]
$\mathcal{MET}$	*metric spaces and uniformly continuous functions;*
$CMET$	*complete metric spaces and uniformly continuous functions;*
$\mathcal{TOPGROUP}$	*topological groups and continuous homomorphisms;*
$\mathcal{AB}_{Loc.Comp}$	*locally compact abelian Hausdorff groups and continuous homomorphisms;*
$\mathcal{NR}_{\mathbb{K}}$	*normed linear spaces and continuous (=bounded) linear maps, ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$);*
$\mathcal{BAN}_{\mathbb{K}}$	*Banach spaces and ccontinuous linear maps, ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$);*
$C^\star - \mathcal{ALG}$	*$C^\star$-algebras and conntinuous involution preserving linear maps;*
$\mathcal{SET}_\star$	*sets with base point (pointed sets) and base point preserving functions;*[7]
$\mathcal{TOP}_\star$	*topological spaces with base point and continuous base point preserving functions;*
$\mathcal{TOP}_{\star\star}$	*bipointed spaces with continuous functions preserving both base points;*
$\mathcal{TOP}_2$	*topological pairs and continuous mappings of pairs;*[8]
$\mathcal{TOP}_n$	*topological n-tupels.*[9]

It is plain that this list could be continued indefinitely. Its notation will be used throughout several examples.

1.2 Abstract Categories

In many instances the attempt to build new categories from old ones by usual algebraic constructions leaves the scope of concrete categories. For this reason there is a more general concept of category not suffering from this restriction.

1.3 Definition (Abstract Category)**.**
An abstract category is a quintuple $C = (\mathcal{O}, \mathcal{M}, \mathrm{dom}, \mathrm{cod}, \circ)$, where

- dom *and* cod *are functions* $\mathcal{M} \longrightarrow \mathcal{O}$*;*
- $\circ$ *is a function* $D := \{(g, f) \in \mathcal{M} \times \mathcal{M} \mid \mathrm{dom}(g) = \mathrm{cod}(f)\} \longrightarrow \mathcal{M}$*,*

such that the following axioms are satisfied:

1. **[Matching condition]**
   $(g, f) \in D \Rightarrow \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \wedge \mathrm{cod}(g \circ f) = \mathrm{cod}(g)$*;*
2. **[Associativity]**
   $(g, f) \in D \wedge (h, g) \in D \Rightarrow (h \circ g) \circ f = h \circ (g \circ f)$*;*

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[6] A space X is completely regular iff all its points are closed and it satisfies the $T3.5$-separation axiom. $T3.5$ claims that, for any closed set $A \subseteq X$ and any point $x \in X \setminus A$, there is a continuous function $f\colon X \longrightarrow [0, 1]$ such that $f(x) = 1$ and $f(A) \subseteq \{0\}$. A completely regular space is also called 'Tychonoff space'.

[7] A pointed set is a pair (X, x_0) with $x_0 \in X$. A base point preserving function $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a map $f \in Y^X$ such that $f(x_0) = y_0$.

[8] Objects are pairs of topological spaces (X, A), where $A \subseteq X$ is a subspace. A morphism $f\colon (X, A) \longrightarrow (Y, B)$ is a continuous map $f\colon X \longrightarrow Y$ with $f(A) \subseteq B$.

[9] Objects are n-tupels $X = (X_1, \dots, X_n)$, where X_1 is a space, and $X_k \subseteq X_1\ \forall k = 1, \dots, n$. A morphism $f\colon X \longrightarrow Y$ is a continuous function $f\colon X_1 \longrightarrow Y_1$ such that $f(X_k) \subseteq Y_k\ \forall k$.

3. **[Identity existence]**
   $\forall a \in \mathcal{O}\ \exists e \in \mathcal{M}\big(\mathrm{dom}(e) = \mathrm{cod}(e) = a \wedge$
   $$\wedge \forall f\, [(f, e) \in D \Rightarrow f \circ e = f\ \dot{\wedge}\ (e, f) \in D \Rightarrow e \circ f = f]\big);$$

4. **[Morphism set condition]**
   $\mathrm{hom}(a, b) := \{f \in \mathcal{M} \mid \mathrm{dom}(f) = a \wedge \mathrm{cod}(f) = b\} \in \mathbf{V}$.

$\mathcal{O}$ is the class of objects, $\mathcal{M}$ is the class of morphisms. The partial function $\mathcal{M} \times \mathcal{M} \supseteq D \overset{\circ}{\longrightarrow} \mathcal{M}$ is called the composition law.

As within concrete categories we write $f\colon a \longrightarrow b$ or $a \overset{f}{\longrightarrow} b$ to indicate that $f \in \mathrm{hom}(a, b)$. We call f a **morphism** with domain a and codomain b, an **arrow** from a to b, or a **map**.

Composition $\circ$ is a binary operation with domain D turning $\mathcal{M}$ into a partial semigroup. Mostly we write gf for $g \circ f$. Whenever a composition gf occurs, it is assumed that $(g, f) \in D$.

When dealing with several categories C, we use a subscript $\mathcal{O}_\mathcal{C}, \mathcal{M}_\mathcal{C}$ etc. When clear from context, $x \in C$ means $x \in \mathcal{O}_\mathcal{C}$ or $x \in \mathcal{M}_\mathcal{C}$ depending on the nature of x.

Note that by Axiom 1, the hypothesis of Axiom 2 is sufficient to assure that its conclusion does make sense. Since

$$\mathrm{dom}(hg) = \mathrm{dom}(g) = \mathrm{cod}(f)$$

we have $(hg, f) \in D$. Furthermore

$$\mathrm{dom}(h) = \mathrm{cod}(g) = \mathrm{cod}(gf)$$

and thus $(h, gf) \in D$. Now

$$\begin{aligned} &\mathrm{dom}(hg \circ f) = \mathrm{dom}(f) = \mathrm{dom}(gf) = \mathrm{dom}(h \circ gf) \quad \text{and} \\ &\mathrm{cod}(hg \circ f) = \mathrm{cod}(hg) = \mathrm{cod}(h) = \mathrm{cod}(h \circ gf). \end{aligned}$$

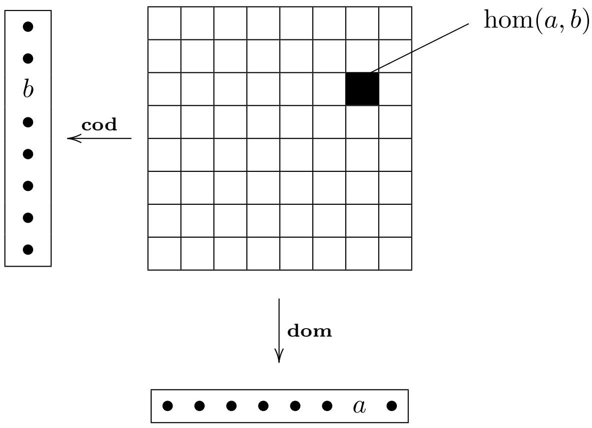
So, by the hypothesis of Axiom 2, it is possible that $hg \circ f = h \circ gf$ and the axiom postulates that this be the case.

Axiom 3 mimics the Concrete Categories' Axiom 2 by forcing the existence of an identity morphism for each object.

Axiom 4 is necessary to assure that constructions are possible within the underlying set-theoretic system.

The functions dom and cod generate a partition of $\mathcal{M}$ into disjoint sets

$$\mathcal{M} = \bigcup_{(a,b) \in \mathcal{O} \times \mathcal{O}} \mathrm{hom}(a, b).$$



In practice an abstract category often appears in a vestment

$$\big(\mathcal{O}, (\hom(a,b))_{(a,b)\in\mathcal{O}\times\mathcal{O}}, \circ\big)\,.$$

The form corresponding to Definition 1.3 is then achieved by declaring $\mathcal{M}$ to be the disjoint union of the sets $\hom(a,b)$. For each $f \in \mathcal{M}$ there is then a unique pair of objects (a,b) with $f \in \hom(a,b)$. $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$ establish the domain and codomain maps. The first of the following examples deals with this situation.

1.4 Example (Abstract Categories)**.**

1. **The Category naturally associated with a Concrete Category** *$(\mathcal{O}, U, hom)$ is*

$$\Big(\mathcal{O}, \bigcup_{(a,b)\in\mathcal{O}\times\mathcal{O}} \{a\}\times\hom(a,b)\times\{b\}, \mathrm{dom}, \mathrm{cod}, \circ\Big).$$

Here $\mathrm{dom}(a,f,b) = a$*,* $\mathrm{cod}(a,f,b) = b$ *and* $(b,g,c)\circ(a,f,b) = (a,gf,c)$*. In view of this construction a concrete category is a special abstract category. Therefore, from now on, when we shall say 'category', we will mean 'abstract category', a concrete category being tacitly understood in the sense of this construction.*

2. **The category $\mathcal{REL}$ of sets and relations**

Object class is $\mathbf{V}$*. For* $a, b \in \mathbf{V}$*,* $\hom(a,b) = \mathcal{P}(a\times b)$*. The composition is the usual product of relations.*

3. **The Category of Topological Bundles** $(\mathcal{TOP}\downarrow\mathcal{TOP})$

The object class consists of all triples (x,π,y)*, where* x, y *are objects in* $\mathcal{TOP}$ *and* $\pi\colon x \longrightarrow y$ *is a continuous map.* $\hom((x,\pi,y),(x',\pi',y'))$ *is the set of pairs of continuous maps* (f,g) *which make the diagram*

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ y & \xrightarrow{g} & y' \end{array}$$

commutative. Composition is componentwise, that is, by pasting together corresponding squares.

4. **Quasi-ordered Classes**

   Let $(X, \leq)$ be a quasi-ordered class (a class equipped with a reflexive and transitive relation). $(X, \leq)$ gives rise to a category whose object class $\mathcal{O}$ is X. For $x, y \in X$, we set

$$\hom(x, y) = \begin{cases} \{(x, y)\} & if\ x \leq y, \\ \emptyset & if\ x \not\leq y. \end{cases}$$

   If $x \leq y$ and $y \leq z$ then the compsition is $(y, z) \circ (x, y) = (x, z)$.

   The sets $\hom(x, y)$ *contain at most one arrow. Any category C with the property*

$$\forall x, y \in \mathcal{O} : \ \mathbf{Card}(\hom(x, y)) \leq 1$$

   can be considered that way.

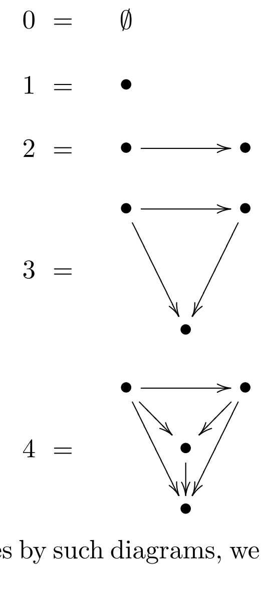
5. **Ordered Classes**

   As an instance of the previous example, partial orders and total orders can be considered as categories.

$$(X, \leq) \text{ is a partially ordered class} \iff \forall x, y \in \mathcal{O} : \ \mathbf{Card}(\hom(x, y) \cup \hom(y, x)) \leq 1;$$
$$(X, \leq) \text{ is a totally ordered class} \iff \forall x, y \in \mathcal{O} : \ \mathbf{Card}(\hom(x, y) \cup \hom(y, x)) = 1.$$

6. **Ordinal Numbers**

   Being well-ordered sets they build a special instance of the last item. Considering their finite members we obtain the positive integers as categories.



   When depicting categories by such diagrams, we omit the requisite identity morphisms.

7. *The class* **On** *of all ordinals is a category.*

8. **Monoids**

   A monoid M defines a category with exactly one object. The set of morphisms is M, composition is given by the product rule of M. Any category with exactly one object is a monoid.

9. **Groups** *As a special instance of the previous point, every group is a category. Cf. Definition 1.54*

10. **The category of matrices $\mathcal{MAT}_\Lambda$**

    Let Λ be a ring. We define the category $\mathcal{MAT}_\Lambda$ of Λ-matrices:

    The class of objects is $\mathbb{N}$. For integers m, n we set

$$\hom(m, n) = \Lambda^{n \times m}$$

    that is, the morphisms are the $n \times m$-matrices with entries in Λ. Composition is the usual matrix product.

11. **The Category ${}_\Lambda\mathcal{GRMOD}$ of $\mathbb{Z}$-graded Λ-Modules**

    $\Lambda \in \mathcal{RING}$. A $\mathbb{Z}$-graded Λ-module is a sequence $(A_n)_{n\in\mathbb{Z}}$ of Λ-modules. For $x \in A_n$ we set $\deg x = n$. A map of degree p ($p \in \mathbb{Z}$) from a graded module A to a graded module B is a sequence of Λ-homomorphisms $f = (f_n)_{n\in\mathbb{Z}}$ with $f_n \colon A_n \longrightarrow B_{n+p}$. For maps $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ of degree p, q respectively, the composition $g \circ f$ is the map of degree $p + q$ given by

$$(g \circ f)_n = g_{n+p} f_n.$$

    Then we obtain

$$\deg f(x) = \deg f + \deg x \text{ and } \deg(g \circ f) = \deg g + \deg f.$$

    Graded Λ-modules together with maps of arbitrary degree build the category ${}_\Lambda\mathcal{GRMOD}$.

12. **The Category ${}_\Lambda\mathcal{MOD}^Z$**

    Objects are graded Λ-modules as before, but morphisms are restricted to maps of degree 0.

13. **The category ${}_\Lambda\mathcal{CC}$ of chain complexes**

    $\Lambda \in \mathcal{RING}$. A chain complex over Λ is a pair (A, ∂) where A is a $\mathbb{Z}$-graded Λ-module and $\partial \colon A \longrightarrow A$ is a map with $\deg(\partial) = -1$ such that $\partial^2 = 0$. The elements of degree n are referred to as n-dimensional chains, ∂ is called the boundary operator of the complex A, or its differential.

    A chain map between chain complexes $(A, \partial), (A', \partial')$ is a map $f \colon A \longrightarrow A'$ of degree 0 such that $\partial' \circ f = f \circ \partial$. This turns the class of chain complexes together with chain maps into a category which is denoted by ${}_\Lambda\mathcal{CC}$. The formula $\partial^2 = 0$ means $\partial_n \partial_{n+1} = 0 : A_{n+1} \longrightarrow A_{n-1}$. It is equivalent to the condition $\ker(\partial_n) \supseteq \operatorname{im}(\partial_{n+1})$.

14. **The Category $\mathcal{GA}$ of group actions**

Its objects are triples (T, G, α) where T, G are groups and $\alpha\colon T \longrightarrow \operatorname{Aut} G$ is a homomorphism of groups. The group action $T \times G \longrightarrow G$ is written as $t \cdot g = \alpha(t)(g)$. The set $\hom((T, G, \alpha), (T', G', \alpha'))$ consists of pairs (φ, f) where $T \xrightarrow{\varphi} T'$ and $G \xrightarrow{f} G'$ are homomorphisms of groups, such that $f(t \cdot g) = \varphi(t) \cdot f(g)$. Composition is componentwise, i.e., if

$$(T, G, \alpha) \xrightarrow{(\varphi,f)} (T', G', \alpha') \xrightarrow{(\psi,g)} (T'', G'', \alpha'')$$

then $(\psi, g) \circ (\varphi, f) = (\psi\varphi, gf)$.

15. **The category MOD of left modules over arbitrary rings**

Objects are paires (Λ, A), where $\Lambda \in \mathcal{RING}$ and $A \in {}_\Lambda\mathcal{MOD}$. A morphism $(\Lambda, A) \longrightarrow (\Sigma, B)$ is a pair (φ, f), where $\varphi\colon \Lambda \longrightarrow \Sigma$ is a morphism of rings, and $f\colon A \longrightarrow B$ is a morphism of abelian groups, such that $f(\lambda \cdot a) = \varphi(\lambda) \cdot f(a)$.

16. **The category RS of ringed spaces**

Objects are paires $(X, \mathcal{O})$, where X is a topological space and $\mathcal{O}$ is a sheaf[10] *of rings on X. A morphism $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair (f, ϕ) consisting of a continuous map $f\colon X \longrightarrow Y$ and a morphism of sheaves $\phi\colon \mathcal{O}_Y \longrightarrow f_\star\mathcal{O}_\mathcal{X}$.*

17. **The category of schemes**

It consists of schemes and morphisms between them. A scheme is a locally ringed space which locally looks like a ring spectrum.

1.5 Proposition. *Let C be a category.*

1. *For each $a \in \mathcal{O}$ there is exactly one morphism $a \xrightarrow{e} a$ satisfying Axiom 3. It is written 1_a, or just 1, and called the* **identity** *of a.*

2. *For $a \in \mathcal{O}$, $\operatorname{End} a$ carries the structure of a monoid. Its units constitute the automorphism group $\operatorname{Aut} a$.*

1.6 Proposition (Characterization of Identities)**.** *Let C be a category. For any $e \in \mathcal{M}_\mathcal{C}$, the following are equivalent:*

1. *$\exists a \in \mathcal{O}_\mathcal{C}$ with $e = 1_a$;*

2. *$\forall f : (f, e) \in D_\mathcal{C} \Rightarrow f \circ e = f$;*

3. *$\forall f : (e, f) \in D_\mathcal{C} \Rightarrow e \circ f = f$.*

The identities of C are exactly the identities[11] of the partial semigroup $(\mathcal{M}, \circ)$. Since, by Proposition 1.5, objects are encoded as identity morphisms, it is possible to give an object free axiomatization of category theory.

[10] See the section on functors and natural transformations.

[11] An identity in a partial semigroup $(M, \circ)$ where $M \times M \supseteq D \xrightarrow{\circ} M$ is an element $e \in M$ with the property $\forall f \in M\ ([(f, e) \in D \to f \circ e = f] \wedge [(e, f) \in D \to e \circ f = f])$.

1.7 Definition (Object free Axiomatization)**.** *A category is a pair* $(\mathcal{M}, \circ)$ *where* $\mathcal{M}$ *is a class and* $M \times M \supseteq D \xrightarrow{\circ} M$ *is a binary partial operation on* $\mathcal{M}$ *subject to the conditions:*

1. $(h, g) \in D \wedge (g, f) \in D \implies (h \circ g, f) \in D \wedge (h, g \circ f) \in D$;
2. $(h \circ g, f) \in D \iff (h, g \circ f) \in D$ *and in this case* $(h \circ g) \circ f = h \circ (g \circ f)$;
3. $\forall f \in \mathcal{M}\ \exists$ identities $e, e' \in \mathcal{M}$ such that $(f, e) \in D \ \wedge\ (e', f) \in D$;
4. $\forall$ *identities* $e, e' \in \mathcal{M}$*, the class* $\{f \in \mathcal{M} \mid (f, e) \in D \wedge (e', f) \in D\}$ *is a set.*

1.8 Theorem. *The two axiomatic descriptions of category theory given by definitions 1.3 and 1.7 are equivalent. If* $(\mathcal{O}, \mathcal{M}, \mathrm{dom}, \mathrm{cod}, \circ)$ *is a category according to Definition 1.3, then* $(\mathcal{M}, \circ)$ *is a category in the sense of Definition 1.7. Conversely, when* $(\mathcal{M}, \circ)$ *is a category according to Definition 1.7, then* $(\mathcal{O}, \mathcal{M}, \mathrm{dom}, \mathrm{cod}, \circ)$*, where* $\mathcal{O}$ *is the class of identities of* $(\mathcal{M}, \circ)$*,* $\mathrm{dom}(f)$ *is the unique identity* e *such that* $(f, e) \in D$*,* $\mathrm{cod}(f)$ *the one with* $(e, f) \in D$*, is a category given in terms of Definition 1.3.*

1.3 Duality

The symmetry inherent to any of the axiomatic descriptions of category theory allows for a simple dualization concept. To each statement $\mathtt{A}$ about data of a category C, one may build the dual statement $\mathtt{A}^\delta$. The process, called dualization, proves its usefulness in two ways: First, to any categorical notion there is a dual notion which, passing to a particular category, may be quite different in character from the original one. Secondly, to each theorem of category theory there is a dual theorem.

We discuss here the elementary concept of dualization as a syntactic process applicable to elementary category theory. To this end we reformulate category theory in first order language.

1.9 Definition. *Elementary category theory is a theory* $\mathcal{T}$ *formulated in first order predicate logic with identity. It is given by the following data.*

1. $\mathcal{T}$ *has two unary predicate symbols* $\mathtt{O},\mathtt{M}$ *and the binary equality symbol* $=$*. For terms* a *and* f *we write* $a \in \mathtt{O}$ *and* $f \in \mathtt{M}$ *to denote the formulas* $\mathtt{O}(a)$ *and* $\mathtt{M}(f)$ *respectively.*
2. $\mathcal{T}$ *has the symbols* `dom`, `cod` *as unary function symbols and* $\circ$ *as binary function symbol. As usual we write* $g \circ f$ *for* $\circ(g, f)$*.*
3. *The axioms of* $\mathcal{T}$ *are the axioms of a first order predicate logical system with equality, enriched by the following proper axioms:*
   (a) $f \in \mathtt{M} \to \mathtt{dom}(f) \in \mathtt{O} \wedge \mathtt{cod}(f) \in \mathtt{O}$
   (b) $g, f \in \mathtt{M} \to g \circ f \in \mathtt{M}$
   (c) $g, f \in \mathtt{M} \to \mathtt{dom}(g) = \mathtt{cod}(f) \to \mathtt{dom}(g \circ f) = \mathtt{dom}(f) \wedge \mathtt{cod}(g \circ f) = \mathtt{cod}(g)$
   (d) $h, g, f \in \mathtt{M} \to \mathtt{dom}(g) = \mathtt{cod}(f) \wedge \mathtt{dom}(h) = \mathtt{cod}(g) \to (h \circ g) \circ f = h \circ (g \circ f)$

$$(e)\ a \in \mathtt{O} \to \exists_{e\in\mathtt{M}}\Big(\mathtt{dom}(e) = \mathtt{cod}(e) = a \\ \wedge\forall_{f\in\mathtt{M}}\big((\mathtt{dom}(f) = a \to f \circ e = f) \wedge (\mathtt{cod}(f) = a \to e \circ f = f)\big)\Big).$$

The dual $\mathtt{P}^\delta$ of a statement $\mathtt{P}$ of $\mathcal{T}$ is defined by induction on the complexity of the expression $\mathtt{P}$:

1.10 Definition.

x^δ	$\equiv$	x	*for*[12] *variables* x
$(\mathtt{dom}(f))^\delta$	$\equiv$	$\mathtt{cod}(f^\delta)$	*for terms* f
$(\mathtt{cod}(f))^\delta$	$\equiv$	$\mathtt{dom}(f^\delta)$	*for terms* f
$(g \circ f)^\delta$	$\equiv$	$f^\delta \circ g^\delta$	*for terms* f,g
$(f \in \mathtt{M})^\delta$	$\equiv$	$f^\delta \in \mathtt{M}$	*for terms* f
$(a \in \mathtt{O})^\delta$	$\equiv$	$a^\delta \in \mathtt{O}$	*for terms* a
$(\neg\mathtt{A})^\delta$	$\equiv$	$\neg(\mathtt{A}^\delta)$	*for formulas* $\mathtt{A}$
$(\mathtt{A}\Box\mathtt{B})^\delta$	$\equiv$	$(\mathtt{A})^\delta\Box(\mathtt{B})^\delta$	*for formulas* $\mathtt{A},\mathtt{B}$ *and logical connectives* $\Box$
$(\forall_x\mathtt{A})^\delta$	$\equiv$	$\forall_x\left(\mathtt{A}^\delta\right)$	*for formulas* $\mathtt{A}$
$(\exists_x\mathtt{A})^\delta$	$\equiv$	$\exists_x\left(\mathtt{A}^\delta\right)$	*for formulas* $\mathtt{A}$.

Passing from $\mathtt{P}$ to $\mathtt{P}^\delta$ is called **dualization**.

1.11 Example. *Let* $\mathtt{P}$ *be the statement*[13]

$$f \in \mathtt{M} \wedge \forall_{h,k}(h \in \mathtt{M} \wedge k \in \mathtt{M} \wedge \mathtt{cod}(h) = \mathtt{dom}(f) \wedge \mathtt{cod}(k) = \mathtt{dom}(f) \to \\ f \circ h = f \circ k \to h = k).$$

Then its dual $\mathtt{P}^\delta$ *is*

$$f \in \mathtt{M} \wedge \forall_{h,k}(h \in \mathtt{M} \wedge k \in \mathtt{M} \wedge \mathtt{dom}(h) = \mathtt{cod}(f) \wedge \mathtt{dom}(k) = \mathtt{cod}(f) \to \\ h \circ f = k \circ f \to h = k).$$

1.12 Theorem (Principle of Duality)**.** *Let* $\mathtt{P}$ *be a statement of* $\mathcal{T}$*. Then* $\mathtt{P}$ *is a theorem of* $\mathcal{T}$ *if and only if* $\mathtt{P}^\delta$ *is a theorem of* $\mathcal{T}$*.*

Proof. If $\mathtt{P}$ is a theorem of $\mathcal{T}$ then there is a proof π of $\mathtt{P}$. This is a finite sequence of formulas of $\mathcal{T}$ each of which either is an axiom or associated by a logical inference rule to formulas prior in the list. Since the axioms are self-dual and the metafunction δ cuts through logical inference rules, applying δ to each individual formula, we obtain a proof π^δ of $\mathtt{P}^\delta$. Now $\mathtt{P}^{\delta^\delta} \equiv \mathtt{P}$. So dualizing a proof of $\mathtt{P}^\delta$ gives back a proof of $\mathtt{P}$. □

During its development a formal theory frequently extends its language and enlarges its supply of axioms. For instance, in Proposition 1.5, for each object the unique existence of identity morphisms is stated. Therefore a new unary function symbol 1 is introduced, intended to denote this one morphism dependent on objects. Formally this amounts to introducing a new proper axiom, expressing as a rule the behavior of identity morphisms. It is obvious that the process of dualization and the principle of duality apply to all such conservative extensions of the original theory $\mathcal{T}$. Consider the following example:

---

[12] We use the symbol '$\equiv$' to indicate identity in the meta language.

[13] $\mathtt{P}$ expresses the notion of monomorphism. Its dual $\mathtt{P}^\delta$ describes epimorphisms.

1.13 Example. *Let* $\mathtt{Q}$ *be the statement*[14]

$$f \in \mathtt{M} \wedge \exists g \left(g \in \mathtt{M} \wedge \mathtt{dom}(g) = \mathtt{cod}(f) \wedge \mathtt{cod}(g) = \mathtt{dom}(f) \wedge g \circ f = 1_{\mathtt{dom}(f)} \right) .$$

Then statement $\mathtt{Q}^\delta$ *is*

$$f \in \mathtt{M} \wedge \exists g \left(g \in \mathtt{M} \wedge \mathtt{dom}(g) = \mathtt{cod}(f) \wedge \mathtt{cod}(g) = \mathtt{dom}(f) \wedge f \circ g = 1_{\mathtt{cod}(f)} \right) .$$

In the sequel we will formulate several elementary statements of $\mathcal{T}$ in usual relaxed notation. Each of these has a dual statement which can easily be produced from the original. We therefore will in general write down only one of the duals.

1.4 Special Morphisms and Objects

1.14 Definition. *Let A be a category.*

1. *A **section** in A is a morphism having a left invers, i.e.,*

   $$x \xrightarrow{f} y \text{ is a section } \iff \exists r \colon y \longrightarrow x \text{ such that } rf = 1_x.$$

2. *Dually, f is a **retraction** iff it has a right inverse.*[15] *An object x is a **sect** provided that $\exists$ a section $x \longrightarrow y$, i.e., x is the domain of a section. y is a **retract** iff it is the codomain of a retraction.*

3. *f is an **isomorphism** iff it is simultaneously a section and a retraction. In this case the left- and the right inverse of f agree to a unique two-sided inverse f^{-1}.*

1.15 Definition. *Let a, b be objects in a category A. We say that a and b are* **isomorphic***, denoted $a \cong b$, iff there exists an isomorphism $f \colon a \longrightarrow b$. The property of being isomorphic defines an equivalence relation on the class $\mathcal{O}_{\mathcal{A}}$.*

1.16 Definition.

1. *$x \xrightarrow{f} y$ is called a **monomorphism** (also* monic *or* mono*), iff it can be cancelled from the left:*

   $$\text{if } \bullet \overset{h}{\underset{k}{\rightrightarrows}} x \xrightarrow{f} y \text{ such that } fh = fk, \text{ then } h = k.$$

2. *Dually, $x \xrightarrow{f} y$ is an **epimorphism** (also* epic *or* epi*) iff it can be cancelled from the right:*

   $$x \xrightarrow{f} y \overset{h}{\underset{k}{\rightrightarrows}} \bullet \text{ with } hf = kf \implies h = k.$$

3. *A **bimorphism** is an arrow that is both, monic and epic.*

[14] $\mathtt{Q}$ describes the property of f being a section. $\mathtt{Q}^\delta$ means 'f is a retraction'. Cf. the next paragraph.

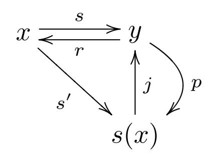
[15] Often a retraction is referred to as 'split epimorphism'.

Often we write $\bullet \overset{f}{\twoheadrightarrow} \bullet$ in order to emphasize that f is epi. The pictogram $\bullet \overset{f}{\rightarrowtail} \bullet$ indicates that f is mono.

Plainly every section is monic, and every retraction is epic. Consequently, each isomorphism is a bimorphism. In a concrete category we have in addition:

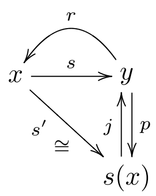
- f section $\Longrightarrow$ f injective $\Longrightarrow$ f monomorphism;
- f retraction $\Longrightarrow$ f surjective $\Longrightarrow$ f epiomorphism.

In the category $\mathcal{TOP}$ consider $x \underset{r}{\overset{s}{\rightleftarrows}} y$ with $rs = 1$. Write s' for the map $x \longrightarrow s(x)$, $\xi \mapsto s(\xi)$, let $s(x) \overset{j}{\longrightarrow} y$ denote inclusion and let $p = s'r$.

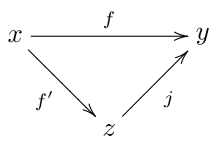


Then $s'rjs' = s'rs = 1s'$. The map s' beeing surjective is epi whence $s'rj = 1$. Also $rjs' = 1$, that is, s' is an isomorphism. In addition $pj = 1$, i.e., $p|s(x) = 1$. When we call a subspace $b \subseteq y$ a topological retract of y iff $\exists p\colon y \longrightarrow b$ with $p|b = 1$, then the section $x \overset{s}{\longrightarrow} y$ is an embedding (i.e. an isomorphism) onto a retract of y, while the retraction $x \overset{r}{\longleftarrow} y$ projects onto that retract.

Conversely, if s is an embedding onto a retract and $r = (s')^{-1}p$, then we obtain $s'rs = s'(s')^{-1}ps = pjs' = s'$ and therefore $rs = 1$.



This observation is in force for all categories whose morphisms $x \overset{f}{\longrightarrow} y$ allow for a factorization



where f' is an epimorphism (cf. Definition 1.39). This is true e.g. for the categories $\mathcal{SET}, \mathcal{TOP}$, ${}_{\Lambda}\mathcal{MOD}$, $\mathcal{AB}, \mathcal{SGRP}, \mathcal{MON}, \mathcal{GROUP}, \mathcal{RING}, \mathcal{ID}$, $\mathcal{ALG}_K$, $\mathcal{CALG}_k$. Observe that for ${}_{\Lambda}\mathcal{MOD}$ this says that a section is just an embedding onto a direct summand, since $x \underset{r}{\overset{s}{\rightleftarrows}} y$ with $rs = 1$ implies that $y = \ker r \oplus \operatorname{im} s$. Among others this is embodied in the following list of examples.

1.17 Example.

1. *If $X \neq \emptyset$ then $f\colon X \longrightarrow Y$ is a section in $\mathcal{SET}$ iff f is injective.*

2. *$f\colon X \longrightarrow Y$ is a section in $\mathcal{TOP}$ iff f is a topological embedding, i.e., f induces a homeomorphism $X \longrightarrow f(X)$ and $f(X)$ is a topological retract* [16] *of Y.*

3. *$f\colon A \longrightarrow B$ is a section in ${}_\Lambda\mathcal{MOD}$ iff f is injective and $\mathrm{im} f$ is a direct summand*[17] *in B.*

4. *Let C be one of $\mathcal{SET}, \mathcal{TOP}, \mathcal{GROUP}, {}_\Lambda\mathcal{MOD}$. Let $X \xrightarrow{f} Y$ be a morphism in C. Then the graph embedding $X \longrightarrow X \times Y, x \mapsto (x, fx)$ is a section in C.*

5. *A morphism in $\mathcal{SET}$ is a retraction if and only if it is surjective.*

6. *$X \xrightarrow{f} Y$ is a retraction in $\mathcal{TOP}$ if and only if there is a continuous retraction r mapping X onto a subspace S of X (and leaving points in S fixed) and a homeomorphism $S \xrightarrow{h} Y$ such that $hr = f$.*

7. *$A \xrightarrow{f} B$ is a retraction in ${}_\Lambda\mathcal{MOD}$ if and only if there is a projection p of A onto a submodule $S \subseteq A$ (with $p^2 = p$ as a map $A \longrightarrow A$) and an isomorphism $S \xrightarrow{h} B$ such that $hp = f$.*

8. *Isomorphisms in $\mathcal{SET}, \mathcal{TOP}, \mathcal{GROUP}, {}_\Lambda\mathcal{MOD}$ are bijections, homeomorphisms, isomorphisms of groups, isomorphisms of modules respectively.*

9. *In $\mathcal{SET}$, $\mathcal{TOP}$, $\mathcal{GROUP}$, $\mathcal{AB}$, ${}_\Lambda\mathcal{MOD}$, $\mathcal{POS}$, $\mathcal{TOP}_{Comp.T2}$ the monomorphisms (epimorphisms) are precisely those morphisms which are injective (surjective) on underlying sets.*[18]

10. *In $\mathcal{TOP}_{T2}$ and $\mathcal{TOP}_{C.reg}$ the monomorphisms are the injective continuous functions, while the epimorphisms are the continuous functions with dense image.*

11. *In $\mathcal{FIELD}$ every morphism is mono.*

12. *$e^{it}\colon (\mathbb{R}, 0) \longrightarrow (S^1, 1)$ is a monomorphism in the category of pointed connected spaces and continuous base-point preserving functions.*

13. *The map $\mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z}$ is a bimorphism in the category of divisible abelian groups. If $D \overset{h}{\underset{k}{\rightrightarrows}} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z}$ with $\pi h = \pi k$, then $h - k\colon D \longrightarrow \mathbb{Z}$ whence $h = k$.*

14. *The embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a bimorphism in $\mathcal{RING}$ and in $\mathcal{MON}$.*

[16] Remember that $A \subseteq Y$ is a topological retract, iff there is a continuous map $r\colon Y \longrightarrow A$ that leaves each point of A fixed.

[17] In analogy to the situation in $\mathcal{TOP}$, a retract of a module M is a submodule N such that $\exists r\colon M \longrightarrow N$ with $r|N = \mathrm{id}_N$. Now we have: $f\colon A \longrightarrow B$ is a section in ${}_\Lambda\mathcal{MOD}$ iff f is injective and $\mathrm{im} f$ is a retract of B.

[18] Cf. Corollary 1.19.

The proofs of these statements are of diverse severity. The assertions concerning monomorphisms are in general obvious. If e.g. $a \xrightarrow{f} b$ is a monomorphism in $\mathcal{RING}$ then the free ring provides an appropriate tool: From $f(x) = f(y)$

$$\mathbb{Z}[t] \underset{y}{\overset{x}{\rightrightarrows}} a \xrightarrow{f} b$$

we get $x = y$. This works in all concrete categories which admit a free object on at least one generator.

Also it is easy to see that in the category ${}_\Lambda\mathcal{MOD}$ the epimorphisms are precisely the surjections: If $a \xrightarrow{f} b$ is epi, and $\pi \colon b \longrightarrow b/\operatorname{im} f$ is the projection, then $\pi f = 0 = 0f$

$$a \xrightarrow{f} b \underset{0}{\overset{\pi}{\rightrightarrows}} b/\operatorname{im} f$$

whence $\pi = 0$, which means that f is surjective.

We just used the cokernel $b/\operatorname{im} f$ as a test object. Proving the corresponding statement for the categories $\mathcal{SET}$ or $\mathcal{TOP}$ works along equal lines.

The proof for the category $\mathcal{GROUP}$ is less obvious. Regarding an epimorphism $a \xrightarrow{f} b$, let k be the normal subgroup of b generated by $f(a)$. Then $\pi f = 0 = 0f$

$$a \xrightarrow{f} b \underset{0}{\overset{\pi}{\rightrightarrows}} b/k$$

but this only gives $b = k$, which does not prove surjectivity of f. Using the cokernel similar to the case of modules fails. One has to choose another test object.

1.18 Lemma. *Let $H < G$ in $\mathcal{GROUP}$.*[19] *Then $\exists K \in \mathcal{GROUP}$ $\exists r, s \colon G \longrightarrow K$ with $r \neq s \wedge r|H = s|H$.*

Proof. Let $2 = \{\pm 1\}$ be the group $\mathbb{Z}_2$ written multiplicatively. $A := 2^G$. We turn A into a G-module with action

$$g \cdot a = a \circ \rho_g$$

where $\rho \colon G^{\mathrm{op}} \longrightarrow S(G)$, $\rho_g(x) = xg$ is the right regular representation.[20] Then

$$(g \cdot a)(x) = a(\rho_g(x)) = a(xg).$$

Now let $K = A \rtimes G$, and consider the split exact sequence

$$1 \longrightarrow A \longrightarrow A \rtimes G \underset{s}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 1$$

$s \colon G \longrightarrow A \times G$ is the splitting homomorphism $s(g) = (e, g)$, where e is neutral in A. Next consider the characteristic function of H:

$$\chi(x) = \begin{cases} +1 & \ldots\ x \in H, \\ -1 & \ldots\ x \in G \setminus H. \end{cases}$$

[19]The expression $H < G$ means that H is a proper subgroup of G.

[20]G^{op} is the opposite group to G, i.e. G with reversed multiplication. $S(G)$ denotes the symmetric group of the set G.

Then $\chi \in A$ and the principal derivation

$$\varphi\colon G \longrightarrow A,\ g \mapsto [g,\chi] = (g\cdot\chi)\chi^{-1}.$$

obeys the cocycle identity

$$\varphi(xy) = \varphi(x)\big(x \cdot \varphi(y)\big)$$

whence the association $g \mapsto (\varphi(g), g)$ defines a homomorphism of groups

$$t\colon G \longrightarrow A \rtimes G.$$

For arbitrary $x \in G$

$$s(x) = t(x) \iff \forall y \in G:\ \chi(yx) = \chi(y).$$

Therefore $s(x) = t(x) \iff x \in H$. Consequently $s|H = t|H$ and $s \neq t$ since H is a proper subgroup. □

1.19 Corollary. *In the category $\mathcal{GROUP}$ the epimporphisms are exactly the surjective group homomorphisms.*

Proof. Let $f\colon X \longrightarrow G$ be an epimorphism in $\mathcal{GROUP}$. If $f(X) < G$, there would be a group K and distinct morphisms $s, t\colon G \longrightarrow K$ with $sf = tf$, which is impossible. □

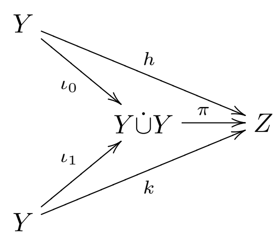
The remarks preceding the last corollary apply similarly to the category of Hausdorff spaces. If they are compact then everything comes right:

For an epimorphism $X \xrightarrow{f} Y$ in $\mathcal{TOP}_{Comp.T2}$ the set $f(X)$ is closed in Y. If f were not surjective, then taking a point $y \in Y \setminus f(X)$, the Urysohn Lemma would provide a continuous function $h\colon Y \longrightarrow [0,1]$ with $h(y) = 1$ and $h(f(X)) = \{0\}$. Then $hf = 0f$ and we would have derived the contradiction $h = 0$.

Also for completely regular spaces these arguments work. In $\mathcal{TOP}_{T2}$ they do not. Also $Y/\overline{f(X)}$ - the natural choice for a test object - is not guaranteed to be a Hausdorff space.[21]

1.20 Lemma. *Let $Y \in \mathcal{TOP}_{T2}$ and $A \subseteq Y$ a closed subspace. Let Z denote the space obtained by glueing together two copies of Y along the points of A. Then $Z \in \mathcal{TOP}_{T2}$.*

Proof. Consider the disjoint union $Y\dot\cup Y = Y\times\{0\}\cup Y\times\{1\}$. Then $Z = Y\dot\cup Y/R$ where R is the equivalence relation generated by $\{((a,0),(a,1)) \mid a \in A\}$. Let $Y \overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}} Y\dot\cup Y$ be the natural embeddings and set $h = \pi\iota_0$, $k = \pi\iota_1$. Thus $h(y) = [y,0]$ and $k(y) = [y,1]$.



[21] Y could e.g. be a non - T3 space.

Consider distinct points $z_1, z_2 \in Z$. We shall construct disjoint neighborhoods of z_1, z_2:

Set $G = h(Y \setminus A)$, $H = k(Y \setminus A)$, $K = h(A) = k(A)$. Then $Z = G \cup K \cup H$ is a partition of Z into the closed subset K and two open sets G, H. If $z_1 \in G$ and $z_2 \in H$ then G, H provide disjoint neigborhoods. If $z_1, z_2 \in h(Y)$, $z_1 = h(y_1)$, $z_2 = h(y_2)$, then choose disjoint open sets $U_1, U_2 \subseteq Y$ with $y_1 \in U_1$ and $y_2 \in U_2$. Then $h(U_1) \cup k(U_1)$ and $h(U_2) \cup k(U_2)$ are open disjoint neighborhoods of z_1, z_2 respectively. □

1.21 Corollary. *In $\mathcal{TOP}_{T2}$ the epimorphisms are precisely the continuous functions with dense image.*

Proof. Let $X \xrightarrow{f} Y$ in $\mathcal{TOP}_{T2}$. If $\overline{f(X)} = Y$ and $Y \underset{k}{\overset{h}{\rightrightarrows}} W$ are such that $hf = kf$, then $h|f(X) = k|f(X)$. Since W is Hausdorff it follows that $h|\overline{f(X)} = k|\overline{f(X)}$, i.e. $h = k$, and therefore f is epi.

Conversely, assume that f is epi. In the notation of the previous lemma set $A = \overline{f(X)}$. Then

$$X \xrightarrow{f} Y \underset{k}{\overset{h}{\rightrightarrows}} Z$$

$hf = kf$ and therefore $h = k$. This says that $[y, 0] = [y, 1]\ \forall y \in Y$ whence $Y = \overline{f(X)}$.

□

The following statements are immediate consequences of the definitions.

1.22 Proposition. *Let A be an arbitrary category, $a \xrightarrow{f} b \xrightarrow{g} c$ in A. Then*

f, g section $\Rightarrow gf$ section;	*f, g retraction $\Rightarrow gf$ retraction.*
f, g monol $\Rightarrow gf$ mono;	*f, g epi $\Rightarrow gf$ epi.*
gf section $\Rightarrow f$ section;	*gf retracton $\Rightarrow g$ retraction.*
gf mono $\Rightarrow f$ mono;	*gf epi $\Rightarrow g$ epi.*
f section $\Rightarrow f$ mono;	*f retraction $\Rightarrow f$ epi.*

1.23 Corollary. *Let $a \xrightarrow{f} b \xrightarrow{g} c$ in A. Then*
f, g bi $\Rightarrow gf$ bi.
f, g iso $\Rightarrow gf$ iso and $(gf)^{-1} = f^{-1}g^{-1}$.
gf iso $\Rightarrow gf$ bi $\Rightarrow f$ mono and g epi.
f iso $\Rightarrow f$ bi.

1.24 Proposition. *In any category the following statemens are equivalent:*

1. *f is an isomorphism.*
2. *f is monomorphism and retraction.*
3. *f is section and epimorphism.*

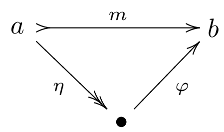
Proof. Assume that $a \xrightarrow{f} b$ is a monic retraction. Then $\exists s$ with $fs = 1_b$. Therefore $f1_a = 1_b f = fsf$. Consequently $sf = 1_a$

$$a \underset{sf}{\overset{1_a}{\rightrightarrows}} a \xrightarrow{f} b$$

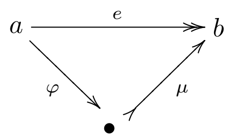
that is, f is an isomorphism. □

1.25 Definition.

- *A monomorphism* $a \xrightarrow{m} b$ *is called* **extremal monomorphism**, *provided that, for any factorization* $m = \varphi\eta$, *if* η *is an epimorphism then* η *is an isomorphism.*



- *An epimorphism* $a \xrightarrow{e} b$ *is called* **extremal epimorphism**, *provided that, for any factorization* $e = \mu\varphi$, *if* μ *is mono then* μ *is an isomorphism.*



1.26 Example.

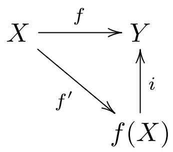
1. *In the categories* $\mathcal{SET}, \mathcal{GROUP}, {}_{\Lambda}\mathcal{MOD}$ *the extremal monomorphisms (extremal epimorphisms) are exactly the usual monomorphisms (epimorphisms), i.e., the injective (surjective) homomorphisms.*

2. *In* $\mathcal{TOP}$ *the extremal monomorphisms are embeddings. Extremal epimorphisms are the quotient maps.*

3. *In* $\mathcal{TOP}_{T2}$ *extremal monomorphisms are closed embeddings, extremal epimorphisms are quotient maps.*

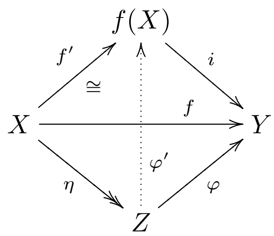
Proof. We verfy the assertions for $\mathcal{TOP}$.

- Consider $X \xrightarrow{f} Y$ in $\mathcal{TOP}$. Assume that f is an extremal monomorphism. Examine the factorization



where i denotes inclusion. f' is epi, thus it must be a homoeomorphism.

Conversely, assume that f is an embedding, and consider a factorization

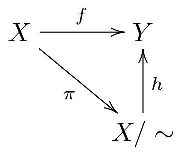


where η epi and thence surjective. Therefore $\varphi(z) = \varphi\eta(x) = f(x) \in f(X)$ which means that φ factors through $f(X)$. We get

$$i\varphi'\eta = \varphi\eta = f = if'$$

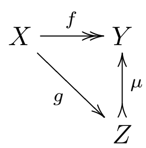
whence $\varphi'\eta = f'$. Since f' is an isomorphism we obtain $(f')^{-1}\varphi'\eta = 1$ which says that η is a section. Proposition 1.24 exposes η as an isomorphism. This demonstrates that f is extremally mono.

- Let $X \xrightarrow{f} Y$ be an extremal epimorphism. Consider the kernel relation $x \sim x' \iff f(x) = f(x')$. f factors through $X/\sim$



and since h is mono it must be a homeomorphism. Consequently f is identifying, i.e., a quotient map.

As to the converse let f be identifying. Examine a factorization



Then μ is a continuous bijection. Let $U \subseteq Z$ be open. Then $f^{-1}(\mu(U)) = g^{-1}(U)$. So $\mu(U)$ has an open f-preimage whence it is open in Y. This shows that μ is a homeomorphism.

□

1.27 Proposition. *For any morphism f the following are equivalent:*

1. *f is an isomorphism.*
2. *f is a monomorphism and an extremal epimorphism.*
3. *f is an extremal monomorphism and an epimorphism.*

Proof. If f is an isomorphism then it is mono and epi. A factorization $f = m\varphi$ entails $1 = m\varphi f^{-1}$, so m is a retraction, hence, if m is mono, then it is iso. This demonstrates that an isomorphism is mono and extremally epi. Likewise it is epi and extremally mono. Conversely the identity $f = 1f$ shows that f is an isomorphism provided that it is extremally mono and epi. □

1.28 Definition.

1. *An object a is called* **initial** *iff for all $x \in \mathcal{O}$ we have* $\mathbf{Card}(\hom(a, x)) = 1$.
2. *Dually: a is* **terminal** *iff* $\forall x \in \mathcal{O} : \mathbf{Card}(\hom(x, a)) = 1$.

3. *a is a* **zero object** *iff it is both, initial and terminal.*

1.29 Example.

1. $\mathcal{SET}, \mathcal{SGRP}, \mathcal{TOP}$*:* $\emptyset$*, the empty set (semigroup, space), is initial. The terminal objects are the singleton sets (semigroups, spaces)* $\{x\}$*.*

2. $\mathcal{RING}$*:* $\mathbb{Z}$ *is initial,* 0 *is terminal.*

3. $\{0,1\}$ *(where* $0 \neq 1$*) is initial in* $\mathcal{BA}$*. The 1-element Boolean algebra* 0 *is terminal in* $\mathcal{BA}$*.*

4. $\mathcal{FIELD}$ *does not own initial or terminal objects, since*

$$\hom(E, F) \neq \emptyset \Rightarrow \operatorname{char} E = \operatorname{char} F.$$

5. *Let* $(X, \leq)$ *be a quasi-ordered class. Then*

$$\begin{array}{rcl} \textit{Initial object} & = & \textit{least element of } X; \\ \textit{terminal object} & = & \textit{greatest element of } X. \end{array}$$

6. 0 *is a zero in* $\mathcal{GROUP}, \mathcal{MON}$*,* $\mathcal{AB}$*,* ${}_{\Lambda}\mathcal{MOD}$*,* $\mathcal{TOPGROUP}$*.*

7. *Singletons are the zeroes in* $\mathcal{TOP}_{\star}$*.*

8. *In a concrete category the initial objects are precisely the free objects on the empty set.*

Plainly any two initial objects are isomorphic. The same holds for terminal objects and 0-objects. In case it exists, the unique[22] zero object is denoted by the symbol 0.

1.30 Definition.

1. *A morphism* $a \xrightarrow{f} b$ *is called* **constant** *provided that for each object* x *and for all* $r, s \in \hom(x, a)$ *we have* $fr = fs$*.*

2. *Dually,* f *is* **coconstant***, iff* $\forall y\ \forall u, v \in \hom(b, y)$ *we have* $uf = vf$*.*

3. f *is a* **zero morphism** *iff it is simultaneously constant and coconstant. We write* 0 *for a 0-morphism, and when necessary,* 0_{ab} *for a 0-morphism* $a \longrightarrow b$*.*

A 0-morphism $a \xrightarrow{0} b$ behaves similar to the number 0 with respect to multiplication. Given the situation

$$\bullet \overset{r}{\underset{s}{\rightrightarrows}} a \xrightarrow{0} b \overset{u}{\underset{v}{\rightrightarrows}} \bullet$$

then $0r = 0s$ and $u0 = v0$.

1.31 Example.

[22] In such a context 'unique' always means 'unique up to isomorphism'.

1. $\mathcal{SET}$ and $\mathcal{TOP}$:

$$x \xrightarrow{f} y \text{ is constant } \iff x = \emptyset \vee \exists \eta \in y : f(x) = \{\eta\}.$$

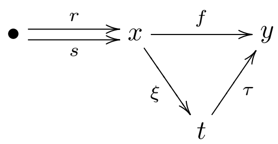
2. *In a concrete category, any morphism which is a constant function in the usual sense on underlying sets is a constant morphism.*

3. *If* $t \in \mathcal{O}_A$ *is terminal then* $\forall x \in \mathcal{O}_A$ *the map* $x \longrightarrow t$ *is constant. Dually, if* $i \in \mathcal{O}_A$ *is initial then* $\forall x \in \mathcal{O}_A$ *the arrow* $i \longrightarrow x$ *is coconstant.*

1.32 Proposition. *Consider* $\bullet \xrightarrow{f} x \xrightarrow{c} y \xrightarrow{g} \bullet$. *If* c *is constant (coconstant, 0-morphism), then so is* gcf.

1.33 Proposition. *Let* A *be a category with terminal object* t. *If* $x \xrightarrow{f} y$ *factors through* t, *then* f *is constant. Conversely, if* f *is constant and* $\hom(t, x) \neq \emptyset$, *then* f *factors through* t.

Proof. If f factors through t, then



$fr = \tau\xi r = \tau\xi s = fs$, that is, f is constant. Conversely, assume that f is constant. Let $u \colon t \longrightarrow x$. Then $f = f1_x = fu\xi$, hence f factors through t. ☐

1.34 Proposition. *Let* $f \colon a \longrightarrow b$ *be a morphism in a category with zero object* 0. *The following statements are equivalent:*

1. f *is constant;*

2. f *is coconstant;*

3. f *is a zero morphism;*

4. f *factors through* 0, *i.e.,* $\exists g, h$ *with* $a \xrightarrow{f} b = a \xrightarrow{g} 0 \xrightarrow{h} b$.

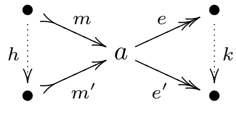
1.35 Proposition. *Let* $f, g \colon a \longrightarrow b$. *Then*

$$f \text{ constant} \wedge g \text{ coconstant} \wedge \hom(b, a) \neq \emptyset \Rightarrow f = g.$$

1.36 Definition. *For an object* a *let* S_a *denote the class of all monomorphisms with codomain* a. *Let* Q_a *be the class of all epimorphims with domain* a.

1. *For* $m, m' \in S_a$ *we write* $m \leq m' \iff \exists h$ *with* $m'h = m$.

2. *For* $e, e' \in Q_a$, $e \geq e' \iff \exists k$ *with* $ke = e'$.



Both relations are reflexive and transitive. We set $m \sim m' \iff m \leq m'$ *and* $m' \leq m$. *Dually,* $e \sim e' \iff e \geq e'$ *and* $e' \geq e$. *The following notions refer to these equivalence relations on* S_a *and* Q_a *respectively.*

- *An equivalence class of monomorphisms in S_a is called a* ***subobject*** *of a. We will write $S(a)$ for the class of all subobjects of a.*

- *An equivalence class of epimorphisms in Q_a is called a* ***quotient object*** *of a. $Q(a)$ denotes the class of all quotient objects of a.*

The class $S(a)$ of all subobjects of an object a is ordered by

$$[m] \leq [n] \iff m \leq n$$

and similarly for quotient objects.[23]

1.37 Definition.

1. *The* ***intersection*** *of a family of subobjects μ_i of a is the infimum $\bigcap_i \mu_i$ in the ordered class $S(a)$.*

2. *Dual notion: The* ***cointersection*** *of a family of quotient objects. This is the infimum in $Q(a)$.*

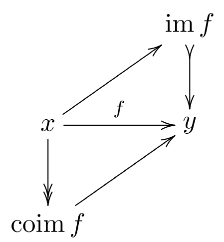
Obviously a monomorphism equivalent to an extremal monomorphism is extremal, and similarly for quotient objects. This allows us to talk about extremal subobjects and extremal quotient objects.

1.38 Definition. *A subobject defined by an extremal monomorpism is called an* ***extremal subobject****. A quotient object defined by an extremal epimorphism is an* ***extremal quotient object****.*

1.39 Definition. *Let $x \xrightarrow{f} y$ be an arrow.*

1. *The* ***image*** *of f is the smallest subobject s of y such that f factors through s. We write $\operatorname{im} f$ for the image of f.*

2. *The* ***coimage*** *of f is the smallest quotient object of x through which f factors. We write $\operatorname{coim} f$ for the coimage of f.*



1.5 Properties of Categories

1.40 Definition. *A category $C = (\mathcal{O}, \mathcal{M}, \mathrm{dom}, \mathrm{cod}, \circ)$ is said to be*

- **small** *provided that C is a set;*

- **discrete** *iff $\mathcal{M} = \{1_a \mid a \in \mathcal{O}\}$;*

[23] Of course one has to define $S(a)$ and $Q(a)$ as a system of representatives wrto. the respective equivalence relation. As it seems closer to ones intuition we will retain the harmless inaccuracy of talking about the 'class of all subobjects (quotient objects)'.

- **strongly connected** *iff* $\forall a, b \in \mathcal{O}: \ \hom(a,b) \neq \emptyset$;
- **connected** *provided that for all objects a, b there is a chain*

$$a = x_0 \longrightarrow x_1 \longrightarrow \cdots\cdots \longrightarrow x_{r-1} \longrightarrow x_r = b,$$

where $x_i \text{ —— } x_{i+1}$ indicates that there is a morphism $x_i \longrightarrow x_{i+1}$ or one $x_i \longleftarrow x_{i+1}$.

A discrete category is simply a class. Therefore a set is a small discrete category.

1.41 Proposition. *The following statements are equivalent.*

• *C is small;*	• $\mathcal{O} \in \mathbf{V}$;	• $\mathcal{M} \in \mathbf{V}$;
• $\mathrm{dom} \in \mathbf{V}$;	• $\mathrm{cod} \in \mathbf{V}$;	• $\circ \in \mathbf{V}$.

In graph theory a **directed multigraph** is a triple (V, E, φ) where V and E are disjoint sets, and $\varphi\colon E \longrightarrow V \times V$. V is the set of vertices, E the set of edges. Thus, a small category A is a special directed multigraph

$$(\mathcal{O}_A, \mathcal{M}_A, \varphi), \quad \text{with} \quad \varphi\colon \mathcal{M}_A \longrightarrow \mathcal{O}_A \times \mathcal{O}_A, \ f \mapsto (\mathrm{dom}(f), \mathrm{cod}(f)).$$

1.42 Proposition. *Any category possessing an initial or a terminal object is connected.*

1.43 Definition. *A category is said to be* **balanced** *provided that each of its bimorphisms is an isomorphism.*

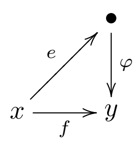
1.44 Example.

1. *$\mathcal{SET}, \mathcal{GROUP}$, ${}_\Lambda\mathcal{MOD}$, $\mathcal{TOP}_{Comp.T2}$ are balanced categories.*
2. *$SGRP, RING, TOP, POS$ are not balanced.*
3. *A partially ordered class is balanced if and only if it is discrete.*

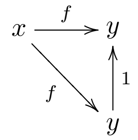
1.45 Proposition. *For any category A the following are equivalent:*

1. *A is balanced.*
2. *Each monomorphism is extremal.*
3. *Each epimorphism is extremal.*

Proof. Assume A being balanced and consider a factorization



where f is mono and e is epi. By Proposition 1.22 e is mono whence it must be an isomorphism. Conversely, if every monomorphism is extremal, the trivial factorization



shows that every bimorphism is an isomorphism. □

1.46 Definition. *A is said to* ***have finite intersections*** *provided that any two subobjects x, y of an arbitrary object possess an intersection $x \cap y$. A* ***has intersections*** *iff arbitrary set-indexed families of subobjects have intersections.*

In many categories appearing in nature the subobjects constitute a *set*.

1.47 Definition. *A category A is called* **well-powered** *provided that each object a has a representative class of monomorphisms which is a set. This means, the class of all subobjects of a is in 1-1 correspondence with a set. Dually, A is called* **co-well-powered** *iff each object a has a set of epimorphisms representing all its quotient objects.*

1.48 Example. *The categories $\mathcal{SET}, \mathcal{GROUP}, {}_\Lambda\mathcal{MOD}, \mathcal{TOP}, \mathcal{TOP}_{T2}$ are well-powered and co-well-powered. The class of ordinal numbers* **On** *is well-powered (by the well-order of* **On***). Since* **On** *is a proper class it is not co-well-powered.*

In a well-powered category we may consider the subobjects of an object a as an ordered set with a greatest element (1_a). Dually, in a co-well-powered category the quotient objects form a set with a greatest element.

1.49 Theorem. *The following statements are equivalent:*

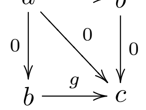
1. $\forall a, b \in \mathcal{O}_\mathcal{C} : hom(a, b)$ *contains a 0-morphism.*
2. $\forall a, b \in \mathcal{O}_\mathcal{C} : hom(a, b)$ *contains exactly one 0-morphism.*
3. $\forall a, b \in \mathcal{O}_\mathcal{C} : hom(a, b)$ *contains exactly one constant.*
4. $\forall a, b \in \mathcal{O}_\mathcal{C} : hom(a, b)$ *contains exactly one coconstant.*
5. $\forall a, b \in \mathcal{O}_\mathcal{C} : hom(a, b)$ *contains a constant and a coconstant.*
6. *There exists a choice function selecting one element out of each* $\hom(a, b)$ *such that the composition of a selected morphism with any morphism (from the left and from the right) is again a selected morphism.*

1.50 Definition. *The category A is* ***pointed*** *provided that each set* $\hom(a, b)$ *contains a 0-morphism 0_{ab}, that is, A satisfies the conditions of Theorem 1.49.*

1.51 Example.

1. *$\mathcal{GROUP}$, $\mathcal{MON}$, ${}_\Lambda\mathcal{MOD}$, $\mathcal{SET}_\star$, $\mathcal{TOP}_\star$ are pointed categories.*
2. *$\mathcal{SET}$, $\mathcal{SGRP}$, $\mathcal{TOP}$, $\mathcal{POS}$, $\mathcal{LAT}$ are not pointed.*

Any two composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in a pointed category provide a commutative diagram



In a pointed category the choice function is uniquely determined and it selects exactly the zero morphisms[24]

$$(0_{ab})_{(a,b)\in A\times A}.$$

Every category with a zero object 0 is pointed, as $a \longrightarrow 0 \longrightarrow b = 0_{ab}$.

1.52 Definition. *A category A is said to be* **skeletal** *provided that*

$$\forall x, y \in A\ (x \cong y \Longrightarrow x = y).$$

1.53 Example.

1. *A class, considered as a discrete category, is skeletal.*
2. *Partially (and totally) ordered classes are skeletal categories.*
3. *If k is a field, then the category of k-matrices $\mathcal{MAT}_k$ is skeletal.*

In a skeletal category all isomorphisms are automorphisms, and distinct objects are never isomorphic. In this respect the following concept is diametrically opposed.

1.54 Definition. *A category is a* ***groupoid*** *iff all its morphisms are isomorphisms.*

1.55 Example. *A group is a groupoid with exactly one object.*

In a groupoid any monoid $\operatorname{End} x$ coincides with $\operatorname{Aut} x$. If there is a morphism $x \xrightarrow{f} y$, then the respective automorphism groups are isomorphic

$$\operatorname{Aut} x \cong \operatorname{Aut} y, \quad u \mapsto fuf^{-1}. \tag{1}$$

1.6 Constructions on Categories

1.56 Definition (Opposition)**.** *Let C be the category $(\mathcal{O}_\mathcal{C}, \mathcal{M}_\mathcal{C}, \mathrm{dom}_C, \mathrm{cod}_C, \circ)$. The opposite (or dual) category to C is*

$$C^{\mathrm{op}} = \big(\mathcal{O}_\mathcal{C}, \mathcal{M}_\mathcal{C}, \mathrm{cod}_C, \mathrm{dom}_C, \star\big)$$

where we define $f \star g = g \circ f$, whenever $g \circ f$ is defined. Obviously $(C^{\mathrm{op}})^{\mathrm{op}} = C$.

1.57 Example.

1. *If C is a group, the opposite category C^{op} is the opposite group as it appears in the proof of Lemma 1.18.*
2. *The opposite category of a partially ordered class $(X, \leq)$ is the inversely ordered class $(X, \geq)$.*

[24]When there is no danger of confusion we will omit the reference to domain and codomain of 0-morphisms, writing them all simply as 0.

The opposite category C^{op} is obtained from C by reverting all arrows. C^{op} allows a semantic view upon dualization:

Given a statement $\mathtt{P}$ about some data of a category C, we may express this statement for the data of C^{op} and then rephrase it in terms of C. The outcome of this process is the dual statement $\mathtt{P}^{\delta}$.

1.58 Definition. *Let A, B be categories. B is a* **subcategory** *of A iff*

- $\mathcal{O}_{\mathcal{B}} \subseteq \mathcal{O}_{\mathcal{A}}$,
- $\mathcal{M}_{\mathcal{B}} \subseteq \mathcal{M}_{\mathcal{A}}$,
- dom_B, cod_B *and composition in B are restrictions of* dom_A, cod_A *and composition in A respectivly,*
- *every B-identity is an A-identity.*

We write $B \subseteq A$ to indicate that B is a subcategory of A. Evidently then

$$\forall x, y \in \mathcal{O}_{\mathcal{B}} : \ \hom_B(x, y) \subseteq \hom_A(x, y).$$

1.59 Example. *Consider a category A that consists of one object $\star$ and two morphisms* $\mathrm{End}_A \star = \{1, x\}$ *with composition rule $xx = x$. Further let B be the category with* $\mathcal{O}_B = \{\star\}$ *and* $\mathrm{End}_B \star = \{x\}$.

A	1	x
1	1	x
x	x	x

B	x
x	x

Due to the last item in Definition 1.58, B is not a subcategory of A. As A and B are just monoids, it is natural that this should happen.

1.60 Definition. *Let $B \subseteq A$.*

1. *B is a* **full subcategory** *of A* $\iff$ $\forall x, y \in \mathcal{O}_{\mathcal{B}} : \hom_B(x, y) = hom_A(x, y)$.
2. *B is a* **dense subcategory** *of A* $\iff$ $\forall a \in \mathcal{O}_{\mathcal{A}} \exists b \in \mathcal{O}_{\mathcal{B}}$ *with* $a \cong b$.
3. *B is* **isomorphism closed** *in A* $\iff$ $\forall b \in B\ \forall a \in A : b \cong a \ \Rightarrow \ a \in B$.

1.61 Example.

1. *The category of finite sets is a full isomorphism closed subcategory of $\mathcal{SET}$.*
2. *The category of sets and injective functions is a subcategory of $\mathcal{SET}$ which is not full.*
3. $\mathcal{AB} \subset \mathcal{GROUP}$ *and* $\mathcal{GROUP} \subset \mathcal{MON}$*, both are full.* $\mathcal{MON} \subset \mathcal{SGRP}$ *is not full.* $\mathcal{GROUP} \subset \mathcal{SGRP}$ *is full.*
4. $\mathcal{BA} \subset \mathcal{LAT} \subset \mathcal{POS}$*; neither is full.*
5. $\mathcal{POS} \subset \mathcal{R}el$ *is full.*
6. $\mathcal{BR} \subset CRING$ *is a full isomorphism closed subcategory.*

7. *By Cayley's theorem, the category of permutation groups (i.e. subgroups of the automorphism groups of the category $\mathcal{SET}$) is a full dense subcategory of $\mathcal{GROUP}$.*

8. *The category of cardinal numbers is a full dense subcategory of $\mathcal{SET}$.*

9. *Let A be an arbitrary category. Any subclass $X \subseteq \mathcal{O}_{\mathcal{A}}$ defines the full subcategory*
$$\Big(X, \bigcup_{x,y\in X} \hom(x,y), \mathrm{dom}, \mathrm{cod}, \circ\Big) \subseteq A.$$

10. *Every full subcategory of a pointed category is pointed.*

1.62 Definition. *A* **skeleton** *of a category A is a maximal full skeletal subcategory S of A.*

1.63 Example.

1. *The powers k^n ($n \in \mathbb{N}$) build a skeleton for the category of finite-dimensional k-vector spaces.*

2. *The class of cardinal numbers is a skeleton for $\mathcal{SET}$.*

3. *Ordinal numbers constitute a skeleton for well-ordered sets and monotone functions.*

1.64 Proposition. *Every category has a skeleton.*[25]

1.65 Definition (Product)**.** *Let A, B be categories. The product of A and B is the category*

$$A \times B = (\mathcal{O}_{\mathcal{A}} \times \mathcal{O}_{\mathcal{B}}, \mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}, \mathrm{dom}_{\mathcal{A}} \times \mathrm{dom}_{\mathcal{B}}, \mathrm{cod}_A \times \mathrm{cod}_B, \circ)$$

where $(g_1, g_2) \circ (f_1, f_2) = (g_1 f_1, g_2 f_2)$. The product of a set-indexed family of categories is defined accordingly.

1.66 Definition (Sum)**.** *The sum of two categories A, B is*

$$A \sqcup B = (\mathcal{O}_A \mathbin{\dot\cup} \mathcal{O}_B, \mathcal{M}_A \mathbin{\dot\cup} \mathcal{M}_B, \mathrm{dom}_A \mathbin{\dot\cup} \mathrm{dom}_B, \mathrm{cod}_A \mathbin{\dot\cup} \mathrm{cod}_B, \circ_A \mathbin{\dot\cup} \circ_B)$$

where the dotted symbol denotes disjoint union. The sum of a set-indexed family of categories is defined in the same way.

1.67 Example. *The small skeletal groupoids are precisely (categorical) sums of groups.*

1.68 Definition (Quotient)**.** *An equivalence relation ρ on the morphism class of a category C is a* **congruence** *provided that*

- $f \rho g \Rightarrow \mathrm{dom}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(f) = \mathrm{cod}(g)$;
- $f \rho f' \wedge g \rho g' \Rightarrow (gf)\rho(g'f')$ *whenever these compositions are defined.*

Given a congruence on C, the **quotient category** *is defined as the category C/ρ, where $\mathcal{O}_{\mathcal{C}/\rho} = \mathcal{O}_{\mathcal{C}}$, $\mathcal{M}_{\mathcal{C}/\rho} = \mathcal{M}_{\mathcal{C}}/\rho$, and composition is given by $[g] \circ [f] = [gf]$.*

[25] That is, what the axiom of choice is good for.

The first point of this definition says that ρ is subordinated to the hom-partition. The second requires compatibility of ρ with composition.

1.69 Example.

1. *Let $X, Y \in \mathcal{TOP}$. A* **homotopy** *is a continuous map $X \times [0,1] \longrightarrow Y$. For continuous maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ we set*

   $$f \simeq g :\Longleftrightarrow \exists\, h\colon X \times [0,1] \longrightarrow Y \text{ with } h(\bullet, 0) = f \ \wedge\ h(\bullet, 1) = g.$$

   We write $h : f \simeq g$ and call h a homotopy from f to g. h describes a continuous deformation from $f(X)$ to $g(X)$; each point $f(x)$ moving along the continuous path $h(x, \bullet)$ towards $g(x)$. The maps f, g are then called homotopic.

   Homotopy of continuous maps is a congruence in $\mathcal{TOP}$. The quotient $\mathcal{TOP}' := \mathcal{TOP}/\simeq$ is the **homotopy category of spaces**. *We denote the set of homotopy classes $\hom_{\mathcal{TOP}'}(X, Y)$ by $[X, Y]$.*

2. *Let $f, g\colon (X, x_0) \longrightarrow (Y, y_0)$ be morphisms in $\mathcal{TOP}_\star$. f and g are homotopic in $\mathcal{TOP}_\star$ iff they are homotopic in $\mathcal{TOP}$ by a homotopy h which for all instances $t \in [0,1]$ defines a morphism $h(\bullet, t)\colon (X, x_0) \longrightarrow (Y, y_0)$ in $\mathcal{TOP}_\star$. So the homotopy h does not move the base point. This defines a congruence in $\mathcal{TOP}_\star$, the respective quotient category $\mathcal{TOP}'_\star$ is called* **homotopy category of pointed spaces**. *The set of morphisms from one pointed space to another is written*

   $$[(X, x_0), (Y, y_0)] = \hom_{\mathcal{TOP}'}((X, x_0), (Y, y_0)).$$

3. *Two morphisms $(X, A) \overset{f}{\underset{g}{\rightrightarrows}} (Y, B)$ in $\mathcal{TOP}_2$ are called* **homotopic** *iff they are homotopic in $\mathcal{TOP}$ via a homotopy h producing morphisms $h(\bullet, t)$ in $\mathcal{TOP}_2$ for all $t \in [0,1]$. Thus the values of points in A must not leave B during deformation. This concept yields a congruence in the category $\mathcal{TOP}_2$. The corresponding quotient is the category ${\mathcal{TOP}_2}'$. We write*

   $$[(X, A), (Y, B)] = \hom_{{\mathcal{TOP}_2}'}((X, A), (Y, B)).$$

4. *The* ***homotopy category of bipointed spaces*** *is $\mathcal{TOP}'_{\star\star} = \mathcal{TOP}_{\star\star}/\simeq$. Homotopy between functions $(X, x_0, x_1) \overset{f}{\underset{g}{\rightrightarrows}} (Y, y_0, y_1)$ is restricted to such continuous maps $h\colon X \times [0,1] \longrightarrow Y$ that produce morphisms $h(\bullet, t)$ in $\mathcal{TOP}_{\star\star}$ $\forall t \in [0,1]$, i.e., $h(x_0, t) = y_0$ and $h(x_1, t) = y_1$. Again the set of morphisms in the quotient category is denoted by brackets.*

5. *Let $(a, \partial), (b, \partial') \in {}_\Lambda\mathcal{CC}$ be chain complexes, $a \overset{f}{\underset{g}{\rightrightarrows}} b$ chain maps. Set*

   $$f \simeq g :\Longleftrightarrow \exists h\colon a \longrightarrow b \text{ in } {}_\Lambda\mathcal{GRMOD} \text{ with } \deg h = 1 \text{ and } \partial' h + h\partial = f - g.$$

We call f and g homotopic provided that $f \simeq g$. The map $h\colon f \simeq g$ is called a **chain homotopy***. Homotopy of chain maps is a congruence in the category ${}_\Lambda\mathcal{CC}$. The resulting quotient ${}_\Lambda\mathcal{CC}' := {}_\Lambda\mathcal{CC}/\simeq$ is the* **homotopy category of chain complexes***. The set of morphisms $[a,b] := \hom_{{}_\Lambda\mathcal{CC}'}(a,b)$ is the abelian group*

$$\hom_{{}_\Lambda\mathcal{CC}}(a,b)/\{f \mid f \simeq 0\}.$$

6. *Let $f, g\colon G \longrightarrow H$ be morphisms in $\mathcal{GROUP}$. Call f and g* **conjugate***, iff they differ by an inner automorphism of H, that is*

$$\exists y \in H\, \forall x \in G: \; g(x) = y f(x) y^{-1} \; .$$

This defines a congruence in $\mathcal{GROUP}$; the resulting quotient category $\mathcal{GROUP}'$ is called the category of **groups and conjugacy classes of homomorphisms***.*

The transition from a category to one of its quotiens may delete certain properties of morphisms.

1.70 Example. *Let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $D = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disk. The embedding $S^1 \hookrightarrow D^2$ is a monomorphism in $\mathcal{TOP}$ while its homotopy class $[f]$ is not mono in $\mathcal{TOP}'$. Similarly, the map $e^{it}\colon \mathbb{R} \longrightarrow S^1$ is epi, while $[e^{it}]$ is not.*

Using homotopy categories we construct a nontrivial groupoid:

1.71 Example. *Let $X \in \mathcal{TOP}$. The* ***fundamental groupoid*** *of the space X is the following small category $\Pi_1(X)$.*

- *Objects of $\Pi_1(X)$ are the points of X.*
- *For points $x, y \in X$ we set*

$$hom_{\Pi_1(X)}(x,y) := [([0,1],0,1),(X,x,y)].$$

Thus, a morphism from x to y is a homotopy class of paths; the homotopy relation being considered in the category of bipointed spaces.

If $\alpha\colon ([0,1],0,1) \longrightarrow (X,x,y)$ and $\beta\colon ([0,1],0,1) \longrightarrow (X,y,z)$ are paths in X ($\alpha(1) = \beta(0)$), then we define the product of α and β as the path

$$\alpha \star \beta\colon ([0,1],0,1) \longrightarrow (X,x,z), \quad (\alpha \star \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Assume that $h\colon \alpha \simeq \alpha'$ and $h\colon \beta \simeq \beta'$ in $\mathcal{TOP}_{\star\star}$. The mappings h and k obey the equations

$$\begin{aligned} h(s,0) &= \alpha(s); & k(s,0) &= \beta(s); \\ h(s,1) &= \alpha'(s); & k(s,1) &= \beta'(s); \\ h(0,t) &= x; & k(0,t) &= y; \\ h(1,t) &= y; & k(1,t) &= z. \end{aligned}$$

For $(s,t) \in [0,1] \times [0,1]$ *set*

$$H(s,t) = \begin{cases} h(2s,t) & 0 \le s \le \frac{1}{2}, \\ k(2s-1,t) & \frac{1}{2} \le s \le 1. \end{cases}$$

The restriction of H *to the closed half-squares* $[0,\frac{1}{2}] \times [0,1]$ *and* $[\frac{1}{2},1] \times [0,1]$ *are continuous and coincide on the overlap, whence* H *is a continuous map* $[0,1] \times [0,1] \longrightarrow X$*. Furthermore*

$$H(s,0) = \left\{ \begin{array}{ll} h(2s,0) & (0 \le s \le \frac{1}{2}) \\ k(2s-1,0) & (\frac{1}{2} \le s \le 1) \end{array} \quad \begin{array}{r} \alpha(2s) \\ \beta(2s-1) \end{array} \right\} = (\alpha \star \beta)(s);$$

$$H(s,1) = \left\{ \begin{array}{ll} h(2s,1) & (0 \le s \le \frac{1}{2}) \\ k(2s-1,1) & (\frac{1}{2} \le s \le 1) \end{array} \quad \begin{array}{r} \alpha'(2s) \\ \beta'(2s-1) \end{array} \right\} = (\alpha' \star \beta')(s).$$

Therefore $H\colon \alpha \star \beta \simeq \alpha' \star \beta'$*. This proves that composition of* $x \xrightarrow{[\alpha]} y$ *and* $y \xrightarrow{[\beta]} z$ *may be correctly defined as*

$$[\beta] \circ [\alpha] = [\alpha \star \beta].$$

Given $x \xrightarrow{[\alpha]} y \xrightarrow{[\beta]} z \xrightarrow{[\gamma]} w$*, one easily constructs a homotopy* $(\alpha \star \beta) \star \gamma \simeq \alpha \star (\beta \star \gamma)$ *whence* $([\gamma] \circ [\beta]) \circ [\alpha] = [\gamma] \circ ([\beta] \circ [\alpha])$*.*

A point x *determines the constant path* $[0,1] \xrightarrow{x} \{x\} \subseteq X$*. Obviously* $x \star \alpha \simeq \alpha$ *and* $\alpha \star y \simeq y$*. Consequently* $[x] = 1_x$ *and* $\Pi_1(X)$ *is a small category. Every morphism* $x \xrightarrow{[\alpha]} y$ *has an inverse, given by*

$$t \mapsto \alpha(1-t).$$

Thus $\Pi_1(X)$ *is indeed a groupoid.*

For each point $x \in X$*, the automorphisms of* x *form a group called the* ***fundamental group*** *of* X *at point* x*, also referred to as* ***first homotopy group***

$$\pi_1(X,x) := \mathrm{Aut}_{\Pi_1(X)}\, x.$$

Points belonging to equal path components have isomorphic fundamental groups (cf. Example 1.55). A path-connected space has therefore a unique fundamental group $\pi_1(X)$*.*

2 Functors

2.1 Definition and Examples

2.1 Definition. *Let* A, B *be categories. A* **functor** *from* A *to* B *is a map*

$$F : \mathcal{M}_A \longrightarrow \mathcal{M}_B$$

that preserves composition and identities. In detail, F *has to satisfy the following rules:*

- *If gf is defined then so is $F(g)F(f)$, and $F(g)F(f) = F(gf)$.*
- *If e is an identity in A then $F(e)$ is an identity in B.*

We write $F\colon A \longrightarrow B$ or $A \xrightarrow{F} B$ when F is a functor from A to B.

A functor $F\colon A \longrightarrow B$ induces a map

$$\mathcal{O}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathcal{B}},\ a \mapsto \mathrm{dom}(F(1_a))$$

which we denote by the same symbol. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_A & \xrightarrow{F} & \mathcal{M}_B \\ {\scriptstyle 1_A}\uparrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathcal{O}_A & \overset{F}{\dashrightarrow} & \mathcal{O}_B \end{array}$$

With this definition[26]we have:

$$F(\hom_A(x,y)) \subseteq \hom_B(Fx, Fy) \text{ and } F(1_x) = 1_{Fx}.$$

The restriction of the map F to a particular hom-set

$$F|\hom_A(x,y)\colon\ \hom_A(x,y) \longrightarrow \hom_B(Fx,Fy)$$

is called a **hom-set restriction** of F.

Since functors are maps on classes they can be composed in the usual way and the result of composing two functors gives again a functor.

$$F : A \longrightarrow B \wedge G : B \longrightarrow C \Rightarrow G \circ F : A \longrightarrow C.$$

We will freely use the notation GF for the composite functor $G \circ F$. A functor $A \longrightarrow A$ is addressed as an **endofunctor**.

2.2 Example (Covariant Functors)**.**

1. *For any category A and subcategory B there is an* **inclusion functor** *$B \hookrightarrow A$. For $A = B$ this is the* **identity functor** *1_A.*
2. *If $\simeq$ is a congruence on A then $f \mapsto [f]$ defines the* **canonical projection** *$A \longrightarrow A/\simeq$.*
3. *If A, B are categories and b is an object in B, there is the* **constant functor**
$$C_b\colon A \longrightarrow B,\ f \mapsto 1_b.$$
4. *Let $A_1, \dots, A_n$ be categories and $a = (a_1, \dots, a_n) \in A_1 \times \cdots \times A_n$ a tuple of objects. For $1 \le i \le n$ the association*
$$x \xrightarrow{f} y \mapsto\ (a_1, \dots, x, \dots, a_n) \xrightarrow{(1,\dots,f,\dots,1)} (a_1, \dots, y, \dots, a_n)$$
defines a functor $\iota_{a,i}\colon A_i \longrightarrow A_1 \times \cdots \times A_i \times \cdots \times A_n$.

[26]We write $F(z)$ or Fz for the value of the functor F at an argument z, no matter whether z denotes a morphism or an object.

5. *For any concrete category C there is functor $U\colon C \longrightarrow \mathcal{SET}$ called the* **underlying functor** *of C. This is a particular example of a* **forgetful functor** *(U forgets additional structure).*

   Similarly there are forgetful functors $\mathcal{RING} \longrightarrow \mathcal{AB}$, $\mathcal{RING} \longrightarrow \mathcal{MON}$ etc. When clear from context what is to be forgotten, we will denote a forgetful functor by $|\bullet|\colon A \longrightarrow B$. That is, $|a|$ is the result of forgetting those parts of the structure of the object $a \in A$ necessary for considering it as an object of B.

6. *There are two obvious functors $\mathcal{LAT} \longrightarrow \mathcal{SGRP}$; one forgets the join, the other one forgets the meet.*

7. *The associaton $X \mapsto F_X$ (where F_X denotes the free group on the set X) constitutes a functor $\mathcal{SET} \longrightarrow \mathcal{GROUP}$, called the* **free group functor**. *Likewise there are free functors for the categories $\mathcal{SGRP}$, $\mathcal{RING}$, $\mathcal{AB}$, ${}_\Lambda\mathcal{MOD}$.*

8. *For any group G denote by G' the commutator subgroup of G (i.e. the group generated by all commutators $xyx^{-1}y^{-1}$ for $x, y \in G$). Since G' is a fully invariant subgroup of G we have a functor $\mathrm{Ab}\colon \mathcal{GROUP} \longrightarrow \mathcal{AB}$ defined by the object map $G \mapsto G/G'$. Ab is called the* **abelianization functor**. *$\mathrm{Ab}(G)$ is the greatest abelian quotient of the group G. Cf. Example 3.3.*

9. *Collapsing subspaces to points defines a functor $\mathcal{TOP}_2 \longrightarrow \mathcal{TOP}_\star$*

$$(X, A) \xrightarrow{\ f\ } (Y, B) \qquad \longmapsto \qquad (X/A, \star) \xrightarrow{\ \tilde{f}\ } (Y/B, \star).$$

10. *There is a functor $\beta\colon \mathcal{TOP}_{C.Reg} \longrightarrow \mathcal{TOP}_{Comp.T2}$ that assigns to each completely regular space X its* **Stone-Čech compactification** *βX.*

11. *The n^{th} projection of graded modules is a functor*

$$\pi_n\colon {}_\Lambda\mathcal{MOD}^{\mathbb{Z}} \longrightarrow {}_\Lambda\mathcal{MOD}.$$

12. *Consider the category of chain complexes ${}_\Lambda\mathcal{CC}$. For each chain complex $A = (A, \partial)$ let $Z(A) = \ker\partial$ and $B(A) = \operatorname{im}\partial$. These are graded modules. The elements of $Z_n(A)$ and $B_n(A)$ are the* ***cycles*** *and* ***boundaries*** *of the complex A in dimension n. It is clear that a chain map $A \longrightarrow B$ throws n-cycles to n-cycles and n-boundaries to n-boundaries. Consequently cycles and boundaries constitute functors*

$$Z\colon {}_\Lambda\mathcal{CC} \longrightarrow {}_\Lambda\mathcal{MOD}^{\mathbb{Z}} \text{ and } B\colon {}_\Lambda\mathcal{CC} \longrightarrow {}_\Lambda\mathcal{MOD}^{\mathbb{Z}}.$$

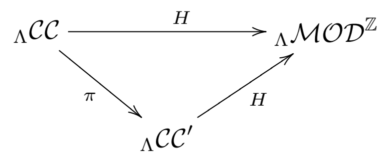
   As $B_n(A) \subseteq Z_n(A)$ for all $n \in \mathbb{Z}$, there is another functor

$$H\colon {}_\Lambda\mathcal{CC} \longrightarrow {}_\Lambda\mathcal{MOD}^{\mathbb{Z}}.$$

   This is the ***homology*** *functor. The n^{th} homology of A is given by composing H with the n^{th} projection. So we have*

$$H_n(A) = (\pi_n \circ H)(A) = Z_n(A)/B_n(A) = \ker\partial_n/\operatorname{im}\partial_{n+1}.$$

13. *Consider chain maps* $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ *in* ${}_{\Lambda}\mathcal{CC}$*. If there is a homtopy* $h\colon f \simeq g$*, then the equality* $\partial' h + h\partial = f - g$ *shows that* $Hf = Hg$*. Consequently the homology functor factors through the homotopy category.*



2.3 Definition. *A* **contravariant functor** $A \longrightarrow B$ *is a functor from* A^{op} *to* B*, or equivalently from* A *to* B^{op}*. To emphasize the distinction, a (usual) functor* $A \longrightarrow B$ *sometimes is called* **covariant***.*

2.4 Example (Contravariant Functors)**.**

1. *For any contravariant functor* $F\colon A \longrightarrow B$ *there are two associated covariant functors* $F^{\star}\colon A^{\mathrm{op}} \longrightarrow B$ *and* ${}^{\star}F\colon A \longrightarrow B^{\mathrm{op}}$ *which are considered as distinct from each other in categorical contexts (e.g. for duality arguments). Viewing them just as classes of ordered pairs, we have* $F = F^{\star} = {}^{\star}F$*. To avoid notational ambiguity we follow the convention to write down only covariant functors. A contravariant functor from* A *to* B *will be designated as (covariant) functor* $A^{\mathrm{op}} \longrightarrow B$ *or* $A \longrightarrow B^{\mathrm{op}}$*.*

   A similar remark applies in general. A functor $F\colon A \longrightarrow B$ *may well be considered as a functor* $A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$*. When it is necessary to distinguish one from another, we will denote the latter by writing* $F^{\mathrm{op}}\colon A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$*.*

2. *For* $X \in TOP$ *let* $\chi(X)$ *denote the characteristic algebra of* X*, that is, the Boolean algebra of clopen sets.* $X \mapsto \chi(X)$ *defines a functor*

$$\chi\colon \mathcal{TOP}^{\mathrm{op}} \longrightarrow \mathcal{BA}.$$

   Its action on morphisms is given by $\chi(f)(B) = f^{-1}(B)$*.*

3. *For a Boolean algebra* $b \in \mathcal{BA}$ *set*

$$S(b) = \{u \in \mathcal{P}(b) \mid u \textit{ is an ultrafilter}\} \cong \hom_{\mathcal{BA}}(b, 2).$$

   For $x \in b$ *let* $B_x := \{u \in S(b) \mid x \in u\}$*. Plainly we have*

$$\bigcup_{x \in b} B_x = S(b) \textit{ and}$$

$$u \in B_x \cap B_y \Longrightarrow u \in B_{x \sqcap y} \subseteq B_x \cap B_y$$

   whence the set $\{B_x \mid x \in b\}$ *is a basis for a topology on* $S(b)$*.* $S(b)$ *with this topology is the* ***Stone space*** *of* b*.*

   The association $b \mapsto S(b)$ *defines a functor with values in Boolean spaces*

$$S\colon \mathcal{BA}^{\mathrm{op}} \longrightarrow \mathcal{BS}.$$

4. *Let $\mathbf{k} \subseteq \mathbf{K}$ be a field extension, $\mathbb{A}^n_{\mathbf{K}}$ the affine space. Call $X \subseteq \mathbb{A}^n_{\mathbf{K}}$ an* **affine algebraic k-set**, *iff it is the set of zeroes of* $\mathbf{k}$*-polynomials*

$$\exists f_1, \dots, f_r \in \mathbf{k}[x_1, \dots, x_n] : X = \{p \in \mathbb{A}^n_{\mathbf{K}} \mid f_1(p) = \cdots = f_r(p) = 0\}.$$

Write $\mathbf{k}[X] = \mathbf{k}[x_1, \dots, x_n]/IX$ for the coordinate ring of X and let $\mathcal{AFF}(\mathbf{K}/\mathbf{k})$ denote the category of affine algebraic $\mathbf{k}$-sets together with regular maps as morphisms. The association $X \longrightarrow \mathbf{k}[X]$ defines a functor

$$\mathcal{AFF}(\mathbf{K}/\mathbf{k})^{\mathrm{op}} \longrightarrow \mathcal{ALG}_{\mathbf{k}}.$$

Often one meets functors $F\colon A_1 \times \cdots \times A_n \longrightarrow B$ defined on a product category. For any tupel of objects $a = (a_1, \dots, a_n) \in A_1 \times \cdots \times A_n$ the composition

$$F \circ \iota_{a,i} = F(a_1, \dots, a_{i-1}, \bullet, a_{i+1}, \dots, a_n)\colon A_i \longrightarrow B$$

is called an **associated (univariate) functor** to F $(1 \leq i \leq n)$.

In particular, a functor defined on a product of two categories is called a **bifunctor**. Thus a bifunctor $F : A \times B \longrightarrow C$ induces for every object $a \in A$ a **right associated functor** $F(a, \bullet)\colon B \longrightarrow C$ and for each $b \in B$ a **left associated functor** $F(\bullet, b)\colon A \longrightarrow C$, cf. Proposition 2.6.

2.5 Example (Bifunctors)**.**

1. *The cartesian product defines a functor $\mathcal{SET} \times \mathcal{SET} \longrightarrow \mathcal{SET}$. To any set a the corresponding right associated functor is*

$$a \times \bullet\colon \mathcal{SET} \longrightarrow \mathcal{SET}, \quad x \xrightarrow{f} y \ \mapsto \ a \times x \xrightarrow{1\times f} a \times y.$$

2. *The tensor product defines a bifunctor ${}_\Lambda\mathcal{MOD}_\Sigma \times {}_\Sigma\mathcal{MOD}_T \longrightarrow {}_\Lambda\mathcal{MOD}_T$. Its left associated functor to the ΣT-bimodule a is*

$$\bullet \otimes_\Sigma a\colon {}_\Lambda\mathcal{MOD}_\Sigma \longrightarrow {}_\Lambda\mathcal{MOD}_T, \quad x \xrightarrow{f} y \ \mapsto \ x \otimes a \xrightarrow{f\otimes 1} y \otimes a$$

and similarly for right associated functors.

3. *For $n+1$ rings $\Lambda_0, \dots, \Lambda_n$, the tensor product is a functor*

$${}_{\Lambda_0}\mathcal{MOD}_{\Lambda_1} \times \cdots \times {}_{\Lambda_{n-1}}\mathcal{MOD}_{\Lambda_n} \longrightarrow {}_{\Lambda_0}\mathcal{MOD}_{\Lambda_n}, \quad (f_1, \dots, f_n) \mapsto f_1 \otimes \cdots \otimes f_n.$$

Given modules $a_j \in {}_{\Lambda_{j-1}}\mathcal{MOD}_{\Lambda_j}$ $(1 \leq j \leq n)$, defines the associated functor

$$a_1 \otimes \cdots \otimes a_{i-1} \otimes \bullet \otimes a_{i+1} \otimes \cdots \otimes a_n\colon {}_{\Lambda_{i-1}}\mathcal{MOD}_{\Lambda_i} \longrightarrow {}_{\Lambda_0}\mathcal{MOD}_{\Lambda_n}.$$

2.6 Proposition. *A bifunctor $F\colon A \times B \longrightarrow X$ produces two families of functors*

$$F(a, \bullet)\colon B \longrightarrow X \ (a \in \mathcal{O}_A) \text{ and } F(\bullet, b)\colon A \longrightarrow X \ (b \in \mathcal{O}_B).$$

They satisfy

1. $F(a, \bullet)(b) = F(\bullet, b)(a)$ $\forall (a, b) \in \mathcal{O}_A \times \mathcal{O}_B$;

2. *if* $a \xrightarrow{\varphi} a'$ *and* $b \xrightarrow{\psi} b'$ *are morphisms in* A, B *respectively, then the diagram*

$$\begin{array}{ccc} F(a,b) & \xrightarrow{F(\varphi,b)} & F(a',b) \\ {\scriptstyle F(a,\psi)}\downarrow & & \downarrow{\scriptstyle F(a',\psi)} \\ F(a,b') & \xrightarrow{F(\varphi,b')} & F(a',b') \end{array}$$

   is commutative.

If, conversely, we have two families of functors

$$h^a\colon B \longrightarrow X \ (a \in \mathcal{O}_A), \quad h_b\colon A \longrightarrow X \ (b \in \mathcal{O}_B)$$

which satisfy

1. $h^a(b) = h_b(a) \ \forall (a,b) \in \mathcal{O}_A \times \mathcal{O}_B$*;*

2. *for arbitrary arrows* $a \xrightarrow{\varphi} a'$, $b \xrightarrow{\psi} b'$ *in* A, B *respectively, the diagram*

$$\begin{array}{ccc} h^a(b) = h_b(a) & \xrightarrow{h_b(\varphi)} & h_b(a') = h^{a'}(b) \\ {\scriptstyle h^a(\psi)}\downarrow & & \downarrow{\scriptstyle h^{a'}(\psi)} \\ h^a(b') = h_{b'}(a) & \xrightarrow{h_{b'}(\varphi)} & h_{b'}(a') = h^{a'}(b') \end{array}$$

   is commutative;

then there is a unique bifunctor $F\colon A \times B \longrightarrow X$ *such that*

$$F(a,\bullet) = h^a \ \forall a \in \mathcal{O}_A \text{ and } F(\bullet,b) = h_b \ \forall b \in \mathcal{O}_B.$$

Proof. The first statement is obvious. For the converse, assume given the two families h^a and h_b. Then

$$F(a,b) := h^a(b), \quad F(\varphi,\psi) := h^{a'}(\psi) \circ h_b(\varphi)$$

gives the desired unique bifunctor. □

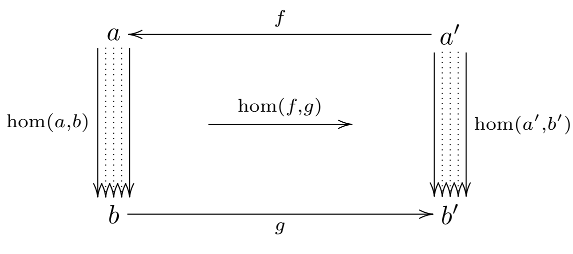
2.2 Hom-Functors

2.7 Definition. *The* ***Hom-functor*** *of a category* C *is the functor*

$$\hom_C : C^{\mathrm{op}} \times C \longrightarrow \mathcal{SET}.$$

It maps each object pair $a, b \in C$ *to the set of all morphisms* $a \longrightarrow b$.

An arrow $(a,b) \longrightarrow (a',b')$ *in* $C^{\mathrm{op}} \times C$ *is a pair* (f,g) *with* $a \xrightarrow{f} a'$ *in* C^{op} *and* $b \xrightarrow{g} b'$ *in* C*. Since in* C^{op} *all arrows are reversed, we are concerned with the diagram*

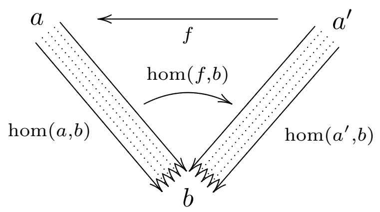


where all indicated arrows are morphisms in C.

The function $\hom_C(f,g)\colon \hom_C(a,b) \longrightarrow \hom_C(a',b')$ *is defined as*

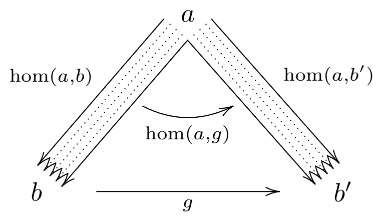
$$\hom(f,g)(\varphi) = g\varphi f.$$

2.8 Definition. *The left associated functor* $\hom_C(\bullet,b) : C^{\mathrm{op}} \longrightarrow \mathcal{SET}$ *is called the* **contravariant hom-functor** *associated to the object b. It may be visualized by the diagram*



The function $\hom(f,b)$ *acts as* $\hom(f,b)(\varphi) = \varphi f$.

Dually, the right associated functor $\hom_C(a,\bullet) : C \longrightarrow \mathcal{SET}$ *to a is the* **covariant hom-functor** *associated to a.*



Now we have $\hom(a,g)(\varphi) = g\varphi$.

2.9 Proposition. *For any category C and object a we have*

$$\hom_C(\bullet,a) = \hom_{C^{\mathrm{op}}}(a,\bullet) \text{ and } \hom_C(a,\bullet) = \hom_{C^{\mathrm{op}}}(\bullet,a).$$

Thus the contravariant hom-functors of C are the corresponding covariant hom-functors of C^{op}. Dually the covariant hom-functors of C coincide with the corresponding contravariant hom-functors of C^{op}.

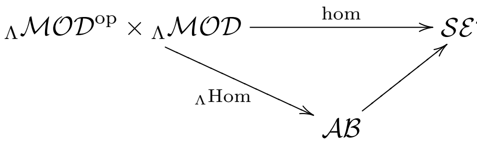
Several properties of objects or morphisms may be rephrased with the aid of hom-functors.

2.10 Proposition. *Let f be a morphism in a category C, and* hom *the functor* $\hom_C\colon C^{\mathrm{op}} \times C \longrightarrow \mathcal{SET}$.

1. *f is mono* $\iff$ $\hom(x,f)$ *is injective* $\forall x \in \mathcal{O}_C$.
2. *f is epi* $\iff$ $\hom(f,x)$ *injective* $\forall x \in \mathcal{O}_X$.
3. *f section* $\iff$ $\hom(f,x)$ *surjective* $\forall x \in \mathcal{O}_C$.
4. *f retraction* $\iff$ $\hom(x,f)$ *surjective* $\forall x \in \mathcal{O}_C$.
5. $\forall x \in \mathcal{O}_C$ $\hom(x,f)$ *bijective* $\iff$ *f iso* $\iff$ $\forall x \in \mathcal{O}_X$ $\hom(f,x)$ *bijective.*

2.3 Internalization

Certain categories allow of endowing their morphism sets with additional structure compatible with composition. For example, the set of morphisms $a \longrightarrow b$ in ${}_\Lambda\mathcal{MOD}$ is an abelian group and composition from the left and from the right is distributive. This can be expressed by saying that the hom-functor of ${}_\Lambda\mathcal{MOD}$ factors through abelian groups.



We shall notationally remind ourselves of this phenomenon by using a deviant symbolism. In case of modules we write $\mathrm{Hom}(a, b)$ instead of $\mathrm{hom}(a, b)$. More precisely, for rings Λ, Σ we set

$$\begin{aligned} {}_\Lambda\mathrm{Hom} &= \mathrm{hom}_{{}_\Lambda\mathcal{MOD}}; \\ \mathrm{Hom}_\Sigma &= \mathrm{hom}_{\mathcal{MOD}_\Sigma}; \\ {}_\Lambda\mathrm{Hom}_\Sigma &= \mathrm{hom}_{{}_\Lambda\mathcal{MOD}_\Sigma}. \end{aligned}$$

We will use the symbol $\star$ for both, ${}_\Lambda\mathrm{Hom}(\bullet, \Lambda)$ and $\mathrm{Hom}_\Lambda(\bullet, \Lambda)$.

2.11 Example (Dual module)**.** *For $a \in {}_\Lambda\mathcal{MOD}$, $a^\star$ is the **dual module** of a. Since Λ is a $\Lambda\Lambda$-bimodule, the dual $a^\star$ is a right Λ-module. If $a \xrightarrow{f} b$ in ${}_\Lambda\mathcal{MOD}$, then $f^\star \colon b^\star \longrightarrow a^\star$ in $\mathcal{MOD}_\Lambda$ and $f^\star(\omega) = \omega f$.*

*Similarly, the dual of a right Λ-module c is the left Λ-module $c^\star = \mathrm{Hom}_\Lambda(c, \Lambda)$. So we can compose the two functors to obtain the **bidual** as an endofunctor on left and on right modules*

$$\begin{aligned} \star\star \colon {}_\Lambda\mathcal{MOD} \xrightarrow{\star} \mathcal{MOD}_\Lambda \xrightarrow{\star} {}_\Lambda\mathcal{MOD} \\ \star\star \colon \mathcal{MOD}_\Lambda \xrightarrow{\star} {}_\Lambda\mathcal{MOD} \xrightarrow{\star} \mathcal{MOD}_\Lambda. \end{aligned}$$

For $a \xrightarrow{f} b$, the map $f^{\star\star} \colon a^{\star\star} \longrightarrow b^{\star\star}$ is given by $f^{\star\star}(\Omega)(\omega) = \Omega(\omega f)$ (cf. Example 3.2).

Equipping - when possible - the hom-sets of a category A in such a way that hom_A factors through A is called **internalization**. It is obvious that this happens with the categories $\mathcal{MOD}_k$ for commutative rings k. It can also be achieved for topological spaces, as we shall see now.

2.12 Example (Compact-Open Topology)**.** *Let $X, Y \in \mathcal{TOP}$. For $A \subseteq X$ and $B \subseteq Y$, we write B^A for the set $\{f \in C(X, Y) \mid f(A) \subseteq B\}$. The **compact-open topology** on $C(X, Y)$, written* `CO`*, is the topology generated as a subbasis by*

$$\{U^K \mid K \subseteq X \text{ compact} \wedge U \subseteq Y \text{ open}\}.$$

The `CO`-topology on $C(X, Y)$ is stronger than the topology on $C(X, Y)$ which is induced by the product topology on $|Y|^{|X|}$: Let $\pi_x \colon C(X, Y) \longrightarrow Y$, $f \mapsto f(x)$ denote evaluation. The set $\{\pi_x^{-1}(U) \mid U \subseteq Y \text{ open } \wedge\ x \in X\}$ is a subbasis for the topology on $C(X, Y)$ induced by the product topology on $|Y|^{|X|}$. Obviously $\pi_x^{-1}(U) = U^{\{x\}}$, hence it is open in `CO`. In case X is discrete, $C(X, Y) = |Y|^{|X|}$ and `CO`=product topology.

2.13 Proposition.

1. *$\forall x \in X$ the evaluation $\pi_x \colon Y^X \longrightarrow Y$, $f \mapsto f(x)$ is continuous;*

2. *$Y \in \mathcal{TOP}_{T2} \;\Rightarrow\; Y^X \in \mathcal{TOP}_{T2}$.*

Proof.

1. Take $f \in C(X,Y)$ and let $U \subseteq Y$ be open with $\pi_x(f) \in U$. Then $U^{\{x\}}$ is an open neighborhood of f and $\pi_x\left(U^{\{x\}}\right) \subseteq U$.

2. If $f \neq g$ then $\exists x \in X$ with $fx \neq gx$. So there are disjoint open sets $U, V \subseteq Y$ with $fx \in U$, $gx \in V$. Therefore $f \in U^{\{x\}}$, $g \in V^{\{x\}}$ and $U^{\{x\}} \cap V^{\{x\}} = \emptyset$.

□

We endowe all hom-sets $C(X,Y)$ of the category $\mathcal{TOP}$ with the CO-topology.

2.14 Theorem. *The hom-functor of $\mathcal{TOP}$ factors through $\mathcal{TOP}$.*

$$\begin{array}{ccc} & \nearrow & \mathcal{TOP} \\ & & \downarrow \\ \mathcal{TOP}^{\mathrm{op}} \times \mathcal{TOP} & \xrightarrow{\text{hom}} & \mathcal{SET} \end{array}$$

Proof. Let $X_1 \xleftarrow{f} Y_1$, $X_2 \xrightarrow{g} Y_2$ be continuous. If $K \subseteq Y_1$ is compact and $U \subseteq Y_2$ is open, then $C(f,g)^{-1}(U^K) = g^{-1}(U)^{f(K)}$. Thus $C(f,g)$ is a continuous map $C(X_1,X_2) \longrightarrow C(Y_1,Y_2)$. □

Since functors with appropriate domain and codomain can be composed, they build a category, as long as they are able to be elements.

2.15 Definition. *Let* **cat** *be the category whose object class consists of all small categories. Morphisms from A to B are the functors $A \longrightarrow B$, composition in* **cat** *is ordinary composition of mappings.*

2.16 Example.

1. *To every space $X \in \mathcal{TOP}$ is associated the groupoid $\Pi_1(X)$, cf. Example 1.71. This assignment defines a functor*

$$\Pi_1 \colon \mathcal{TOP} \longrightarrow \mathbf{cat}.$$

For a continuous map $X \xrightarrow{f} Y$, $\Pi_1(f)$ is a functor $\Pi_1(X) \longrightarrow \Pi_1(Y)$, when we set $\Pi_1(f)(x) = f(x)$ on points $x \in X$ - i.e. on objects of $\Pi_1(X)$ - and $\Pi_1(f)[\alpha] = [f \circ \alpha]$, for $\alpha \colon [0,1] \longrightarrow X$. If $\alpha \simeq \beta$ in $\mathcal{TOP}_{\star\star}$ then $f \circ \alpha \simeq f \circ \beta$, whence $\Pi_1(f)$ is well-defined.

If $x \xrightarrow{[\alpha]} y \xrightarrow{[\beta]} z$ in $\Pi_1(X)$ then $f \circ (\alpha \star \beta) = (f \circ \alpha) \star (f \circ \beta)$, and therefore

$$\Pi_1(f)([\beta] \circ [\alpha]) = [f \circ (\alpha \star \beta)] = [(f \circ \alpha) \star (f \circ \beta)] = \Pi_1(f)[\beta] \circ \Pi_1(f)[\alpha].$$

Since also $\Pi_1(f)(1_x) = \Pi_1(f)[x] = [f \circ x] = [f(x)] = 1_{\Pi_1(f)(x)}$, *it is clear that* $\Pi_1(f)\colon \Pi_1(X) \longrightarrow \Pi_1(Y)$ *is a functor.*

It is plain that $\Pi_1(1_X) = 1_{\Pi_1(X)}$ *and, for continuous maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *that* $\Pi_1(gf) = \Pi_1(g)\Pi_1(f)$. *This demonstrates that* $\Pi_1\colon \mathcal{TOP} \longrightarrow$ **cat** *is a functor.*

2. *As a pointed space has a unique fundamental group, there is the* ***fundamental group functor***

   $$\pi_1\colon \mathcal{TOP}_\star \longrightarrow \mathcal{GROUP}, \quad (X, x_0) \xrightarrow{f} (Y, y_0) \mapsto \pi_1(X, x_0) \xrightarrow{\pi_1(f)} \pi_1(Y, y_0)$$

   where $\pi_1(f) = \Pi_1(f)|\pi_1(X, x_0)$.

The fundamental group functor can also be realized in the homotopy category of (single) pointed spaces.

2.17 Lemma. *Let* $e\colon ([0,1], \{0,1\}) \longrightarrow (\mathbb{S}^1, 1)$ *denote the exponential function,* $e(t) = e^{2\pi i t}$. *Then for any pointed space* Y *the map*

$$\operatorname{Hom}(e, Y)\colon (Y, y_0)^{(\mathbb{S}^1, 1)} \longrightarrow (Y, y_0)^{([0,1], \{0,1\})}$$

is an isomorphism in $\mathcal{TOP}_2$.

Proof. Let $\pi\colon [0,1] \longrightarrow [0,1]/\{0,1\}$. Since $[0,1]$ is compact and $\mathbb{S}^1$ is Hausdorff, the map $e\colon I \longrightarrow \mathbb{S}^1$ is a closed surjection and therefore an identifying map.

$$\begin{array}{ccc} (I, \dot{I}) & \xrightarrow{\quad e \quad} & (\mathbb{S}^1, 1) \\ & {}_{\pi}\searrow \quad \nearrow_{\tilde{e}} & \\ & (I/\dot{I}, \star) & \end{array}$$

Therefore the map $\tilde{e}$ is a homeomorphism $[0,1]/\{0,1\} \cong \mathbb{S}^1$. As e is an epimorphism, the map $\operatorname{Hom}(e, Y)$ is injective.

If $g \in (Y, y_0)^{(I, \dot{I})}$ then $(\mathbb{S}^1, 1) \xrightarrow{\tilde{e}^{-1}} (I/\dot{I}, \star) \xrightarrow{\tilde{g}} (Y, y_0)$ and so $\tilde{g}\tilde{e}^{-1} \in (Y, y_0)^{(\mathbb{S}^1, 1)}$. $\operatorname{Hom}(e, Y)(\tilde{g}\tilde{e}^{-1}) = \tilde{g}\tilde{e}^{-1}\tilde{e}\pi = g$, so $\operatorname{Hom}(e, Y)$ is surjective. so $\operatorname{Hom}(e, Y)$ is a continuous bijection. For $U \subseteq Y$ open and $K \subseteq I$ compact we observe that

$$\operatorname{Hom}(e, Y)\left(U^{e(K)}\right) = U^K.$$

Any compact subset of $\mathbb{S}^1$ can be written as such an $e(K)$ ($C \subseteq \mathbb{S}^1$ compact $\rightarrow$ $C = ee^{-1}(C)$ and $e^{-1}(C) \subseteq I$ is compact). Consequently $\operatorname{Hom}(e, Y)$ maps subbasic open sets of $(Y, y_0)^{(\mathbb{S}^1, 1)}$ to open sets in $(Y, y_0)^{(I, \dot{I})}$. This shows that $\operatorname{Hom}(e, Y)$ is an homeomorphism. □

Using the exponential map $[0,1] \xrightarrow{e^{2\pi i t}} \mathbb{S}^1$ one easily obtains

$$[([0,1], 0, 1), (X, x_0, x_0)] \cong [(\mathbb{S}^1, 1), (X, x_0)].$$

Thus $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$, the product of closed paths being defined accordingly. This point of view leads to the definition of higher homotopy groups.

2.4 Properties of Functors

2.18 Definition. *A functor $F : A \longrightarrow B$ is said to be*

- **faithful** *iff each hom-set restriction $F|\,\hom(x, y)$ is injective.*
- **full** *iff each hom-set restriction is surjective.*
- *an* **embedding** *iff $F\colon \mathcal{M}_A \longrightarrow \mathcal{M}_B$ is injective.*
- **dense** *iff $\forall b \in B\ \exists a \in A$ with $F(a) \cong b$.* [27]

If $F\colon A \longrightarrow B$ is injective (surjective) as a function $\mathcal{M}_A \longrightarrow \mathcal{M}_B$, then the associated object map $\mathcal{O}_A \longrightarrow \mathcal{O}_\mathcal{B}$ is injective (surjective).

In contrast to an embedding, a faithful functor need not be globally injective. Equipping e.g. a certain set x with different topologies τ_1 and τ_2 results in distinct objects $X_1 = (x, \tau_1)$, $X_2 = (x, \tau_2) \in \mathcal{TOP}$. Thus, although the maps 1_{X_1} and 1_{X_2} do not coincide in $\mathcal{TOP}$, the faithful functor $\mathcal{TOP} \longrightarrow \mathcal{SET}$ identifies them.

We may say that a faithful functor is 'locally injective'.

2.19 Example.

1. *Every functor from a category A to a quotient category $A/\simeq$ is full and dense.*
2. *If $A \subseteq B$ then inclusion $A \hookrightarrow B$ is an embedding.*
3. *Forgetful functors are faithful.*
4. *For the following forgetful functors we have:*

$\mathcal{FIELD} \longrightarrow \mathcal{SET}$	*not full, not dense*
$\mathcal{GROUP} \longrightarrow \mathcal{SET}$	*not full, but dense*
$\mathcal{TOP}_{T2} \longrightarrow \mathcal{TOP}$	*full, not dense*

5. *$\mathcal{TOP}_\star \longrightarrow \mathcal{TOP}_2$, $(X, x_0) \mapsto (X, \{x_0\})$ is a full embedding.*
6. *$\mathcal{TOP} \longrightarrow \mathcal{TOP}_2$, $X \mapsto (X, X)$ full embedding.*
7. *$\mathcal{TOP} \longrightarrow \mathcal{TOP}_2$, $X \mapsto (X, \emptyset)$ full embedding.*
8. *$\mathcal{TOP}_n \longrightarrow \mathcal{TOP}_{n+1}$, $(X_1, \ldots, X_n) \mapsto (X_1, \ldots, X_n, \emptyset)$ full embedding. There is a diagram of categories and functors*

$$\begin{array}{ccccccc} \mathcal{TOP} & \longrightarrow & \mathcal{TOP}_2 & \longrightarrow & \mathcal{TOP}_3 & \longrightarrow & \cdots \\ & & \uparrow & & & & \\ & & \mathcal{TOP}_\star & & & & \end{array}$$

each of whose arrows is a full embedding.

[27] In the literature such a functor is also called **representative**.

2.20 Definition. *Let $a \in A$ be an object.*

1. *a is a **separator** iff* $\hom(a, \bullet)$ *is faithful.*
2. *a is a **coseparator** iff* $\hom(\bullet, a)$ *is faithful.*

2.21 Proposition. *a is a separator iff for arbitrary pairs of distinct parallel arrows $x \overset{f}{\underset{g}{\rightrightarrows}} y$ there exists $h\colon a \longrightarrow x$ with*

$$a \xrightarrow{h} x \xrightarrow{f} y \;\neq\; a \xrightarrow{h} x \xrightarrow{g} y.$$

Dually, a is a coseparator iff for arbitrary pairs of distinct parallel arrows $x \overset{f}{\underset{g}{\rightrightarrows}} y$ there exists $h\colon y \longrightarrow a$ with

$$x \xrightarrow{f} y \xrightarrow{h} a \;\neq\; x \xrightarrow{g} y \xrightarrow{h} a.$$

2.22 Example.

1. *In $\mathcal{SET}$, $\mathcal{TOP}$, $\mathcal{RING}$, ${}_\Lambda\mathcal{MOD}$, $\mathcal{AB}$, $\mathcal{MON}$, $\mathcal{SGRP}$, $\mathcal{GROUP}$ the free objects on sets $x \neq \emptyset$ are among the separators.*
2. *In $\mathcal{SET}$, $\mathcal{TOP}$, the separators are exactly the objects with nonempty underlying sets. The coseparators are the sets x with* $\mathbf{Card}(x) \geq 2$.
3. $(\mathbb{N}, +, 0)$ *is a separator in $\mathcal{MON}$.*
4. *$\mathbb{Z}$ is a separator in $\mathcal{GROUP}$ and in $\mathcal{AB}$.*
5. *If k is a field, then the separators and coseparators in $\mathcal{MOD}_k$ both coincide with the nonzero vector spaces.*
6. *In $\mathcal{TOP}$ the coseparators are the non-T0-spaces.*
7. *In $\mathcal{TOP}_{T0}$ the coseparators are precisely the non-T1-spaces.*
8. *The two-element Boolean algebra* 2 *is a coseparator in $\mathcal{BA}$.*
9. $[0, 1]$ *is a coseparator in $\mathcal{TOP}_{C.Reg}$.*
10. $\mathbb{S}^1$ *is a coseparator in both $\mathcal{AB}$ and $\mathcal{AB}_{Loc.Comp}$.*
11. *By the Hahn-Banach Theorem, $\mathbb{C}$ is a coseparator for $\mathcal{BAN}_{\mathbb{C}}$.*
12. *None of $\mathcal{SGRP}$, $\mathcal{GROUP}$, $\mathcal{RING}$, $\mathcal{TOP}_{T2}$ has a coseparator.*
13. *in $\mathcal{POS}$ the separators are the non-empty ordered sets. An object $(x, \leq)$ is a coseparator iff* $\mathbf{Card}(x) \geq 2$ *and $(x, \leq)$ not discrete.*

We check the assertion concerning $\mathcal{POS}$.

If $(a, \leq)$ is a coseparator then $\mathbf{Card}(a) \leq 1$ is impossible. Let 2 denote the ordered set $0 < 1$ and $C_0, C_1\colon 2 \longrightarrow 2$ constant functions. In order that $\hom(\bullet, a)$ detects $C_0 \neq C_1$ we need a non-constant monotone function $2 \longrightarrow a$. Therefore a cannot be discrete.

Conversely assume that $\mathbf{Card}(a) > 1$ and not discrete. Choose $\alpha < \alpha'$ in a and let $x \overset{f}{\underset{g}{\rightrightarrows}} y$ be in $\mathcal{POS}$ such that $f \neq g$. Then $\exists \xi \in x$ with $f\xi \neq g\xi$. Let $<^e$ be a linear extension of the order relation of y. W.l.o.g. $f\xi <^e g\xi$. Then the assignment

$$h\colon y \longrightarrow a,\ \ h\eta = \begin{cases} \alpha \ldots \eta \leq^e f\xi \\ \alpha' \ldots f\xi <^e \eta \end{cases}$$

provides a morphism in $\mathcal{POS}$ such that $hf \neq hg$.

If $A = (\mathcal{O}, U\,\mathrm{hom})$ is a concrete category, then the function $U\colon \mathcal{O} \longrightarrow \mathbf{V}$ can be considered as a functor $A \longrightarrow \mathcal{SET}$ which obviously is faithful. Conversely, if A is an arbitrary category and $F\colon A \longrightarrow \mathcal{SET}$ is a faithful set-valued functor, then $(\mathcal{O}_A, U, h)$ with $U(a) = Fa$ and $h(a,b) = F(\mathrm{hom}_A(a,b))$ defines a concrete category which - considered as an abstract category - is isomorphic to A.

Thus, concrete categories can be considered as pairs (A, F), where A is an (abstract) category and $F\colon A \longrightarrow \mathcal{SET}$ is a faithful set-valued functor.

2.23 Definition. *Let A and B be categories. We say that A is **concretizable over** B iff there is a faithful functor $F\colon A \longrightarrow B$. A category being concretizable over $\mathcal{SET}$ is simply called **concretizable**.*

Plainly, concrete categories are concretizable. But there are several concretizable categories which are not concrete in the original sense.

2.24 Example.

1. *Every small category is concretizable.*
2. *$\mathcal{REL}$ is concretizable via the functor*

$$P\colon \mathcal{REL} \longrightarrow \mathcal{SET}, \quad x \overset{r}{\longrightarrow} y \ \mapsto\ \mathcal{P}(x) \overset{P}{\longrightarrow} \mathcal{P}(y)$$

   with $P(a) = r[a]$.

3. *$\mathcal{SET}^{\mathrm{op}}$ is concretizable. Combining the functor*

$$\mathcal{SET}^{\mathrm{op}} \longrightarrow \mathcal{REL}, \quad f \mapsto f^{-1}$$

   with P gives a faithful functor $\mathcal{SET}^{\mathrm{op}} \longrightarrow \mathcal{SET}$.

4. *A category A is concretizable if and only if A^{op} is so.*
5. *The category of all small categories* **cat** *is concretizable. The natural forgetful functor is*

$$\mathbf{cat} \longrightarrow \mathcal{SET}, \quad A \overset{f}{\longrightarrow} B \mapsto \mathcal{M}_A \overset{f}{\longrightarrow} \mathcal{M}_B.$$

6. *The homotopy category of spaces with base point $\mathcal{TOP}'_\star$ is not concretizable.*

2.25 Definition. *Consider a functor $F : A \longrightarrow B$ and a property P.*

1. *F is said to* **preserve property** *P provided the image under F of each morphism (object, diagram) in A with property P has property P in B.*
2. *$F : A \longrightarrow B$ is said to* **reflect property** *P, if, whenever the image under F of a morphism (object, diagram) has property P in B, then this morphism (object, diagram) must have property P in A.*

2.26 Proposition. *Every functor preserves sections, retractions and isomorphisms.*[28]

2.27 Proposition. *Let A be a strongly connected category with terminal object, B a strongly connected category and $F : A \longrightarrow B$ a functor. Then the following are equivalent:*

1. *F preserves constant morphisms.*
2. *F preserves the terminal object.*

Proof. Let $t \in A$ be terminal. If F preserves constants, then 1_{Ft} is constant in B. For arbitrary $b \in B$ there is then exactly one morphism $u\colon b \longrightarrow Ft$. Conversely, assume that Ft is terminal in B. If $x \xrightarrow{c} y$ is constant in A, then, by Proposition 1.33, c factors through t. Therefore Fc factors through Ft, whence it is constant. □

2.28 Proposition. *Every covariant hom-functor* $\hom(a, \bullet)$ *preserves monomorphisms. Likewise every contravariant hom-functor preserves monomorphisms.*

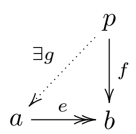
As $\hom(\bullet, b)$ reverses arrows and the dual of mono is epi, the 2nd statement of Proposition 2.28 means: If $x \xrightarrow{f} y$ is an epimorphism in A then $\hom(f, b)$ is a monomorphism $\hom(y, b) \longrightarrow \hom(x, b)$ in $\mathcal{SET}$ i.e., $\hom(f, b)$ is an injective map.

Although all hom-functors preserve monomorphisms, they do not in general preserve their dual. Objects whose hom-functors preserve epimorphisms deserve special attention.

2.29 Definition. *Let $p \in C$ be an object.*

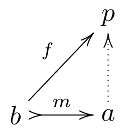
1. *p is called a* **projective object** *in C - or C-projective - iff* $\hom_C(p, \bullet)$ *preserves epimorphisms.*
2. *Dually, p is an* **injective object** *in C, iff it is projective in C^{op}, that is,* $\hom(\bullet, p)$ *preserves epimorphisms.*

C-projectivity of an object p means that for arbitrary epimorphism $a \xrightarrow{e} b$ and any morphism $p \xrightarrow{f} b$ there is some morphism $p \xrightarrow{g} a$ with $eg = f$. We express this by saying that any diagram $a \overset{e}{\twoheadrightarrow} b \xleftarrow{f} p$ can be embedded into a commutative diagram



[28] This is short for the more precise statement: F preserves a morphism's property of being a section, retraction, isomorphism.

Likewise p is injective iff any diagram $p \xleftarrow{f} b \rightarrowtail^{m} a$ can be embedded into a commutative diagram



This allows for the following interpretation:

> p is projective iff each morphism from p to a quotient object of an object a factors through a.
>
> p is injective iff each arrow defined on a subobject of an object a to p extends to a morphism on a.

2.30 Example.

1. *Initial objects are projective. Terminal objects are injective.*

2. *Every set is a projective object in $\mathcal{SET}$. The injective objects in $\mathcal{SET}$ are the nonempty sets.*

3. *An object in ${}_{\Lambda}\mathcal{MOD}$ is projective iff it is a direct summand of a free module.*

4. *Every vector space over a field k is a projective and injective object in $\mathcal{MOD}_k$.*

5. *Let k be a principal ideal domain. As submodules of free modules are free, we have **projective = free** in $\mathcal{MOD}_k$.*

   The injective objects in $\mathcal{MOD}_k$ are precisely the divisible modules.

6. *In $\mathcal{SGRP}, \mathcal{MON}, \mathcal{RING}$ the only inectives are the 0-objects.*

7. *The injective objects in POS are the complete lattices.*

8. *The injective objects in BA are the complete Boolean algebras.*

9. *In $\mathcal{TOP}$, the injective objects are the retracts of powers X^I, where $X = (\{0,1,2\}, \{\emptyset, \{0,1\}, \{0,1,2\}\})$.*

10. *The injective objects in $\mathcal{TOP}_{T0}$ are the retracts of powers S^I, where S is the* **Sierpinski space** $(\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\})$.

11. *The injective objects in $\mathcal{TOP}_{Comp.T2}$ are the retracts of powers $[0,1]^I$. In particular, $[0,1]$ is injective (Tietze-Urysohn Theorem).*

12. *The projective objects in $\mathcal{TOP}_{Comp.T2}$ are the extremally disconnected compact Hausdorff spaces.*

13. *In $\mathcal{POS}, \mathcal{TOP}, \mathcal{AB}, \mathcal{GROUP}$ we have* **projective = free**

2.31 Proposition.

- *A faithful functor reflects:*
  mono-, epi-, bimorphisms, constants, coconstants, zero morphisms, and commutative triangles.

- *A full faithful functor reflects in addition:*
  sections, retractions, isomorphisms.

- *A full faithful dense functor preserves and reflects:*
  mono-, epi-, bimorphisms, constants, coconstants, zero morphisms, sections, retractions, isomorphisms, and commutative triangles.

2.32 Proposition. *Consider functors $F : A \longrightarrow B$ and $G : B \longrightarrow C$.*

- *F and G faithful $\Rightarrow GF$ faithful;*
- *F and G full $\Rightarrow GF$ full;*
- *F and G embedding $\Rightarrow GF$ embedding;*
- *F and G dense $\Rightarrow GF$ dense.*

2.33 Definition. *A functor $F\colon A \longrightarrow B$ is called an* **isomorphism** *provided there is a functor $F^{-1}\colon B \longrightarrow A$ with $F^{-1} \circ F = 1_A$ and $F \circ F^{-1} = 1_B$. Categories A, B are isomorphic, $A \cong B$, iff there exists an isomorphism between them.*

2.34 Proposition. *$F\colon A \longrightarrow B$ a functor. The following are equivalent:*

1. *F is an isomorphism.*
2. *F is bijective (as map $\mathcal{M}_A \longrightarrow \mathcal{M}_B$).*
3. *F is full, faithful and bijective on objects.*

2.35 Example.

1. *Every Boolean algebra can be given the structure of a Boolean ring by definining*

$$x + y := (x \sqcap \overline{y}) \sqcup (y \sqcap \overline{x}) \quad \text{and} \quad x \cdot y := x \sqcap y.$$

   Conversely, each Boolean ring transforms into a Boolean algebra by setting

$$x \sqcap y := xy, \quad x \sqcup y := x + y + xy, \quad \overline{x} := 1 + x.$$

   We therefore have an isomorphism

$$BA \cong BS, \; f \mapsto f.$$

   The association on underlying sets is identity.

2. *Any two skeletons of a category are isomorphic.*

3 Natural Transformations

3.1 Definition and Examples

So far we have seen that categories and functors behave like objects and morphisms in a category. If we ignored set-theoretic complications, we could thus define the 'quasicategory' $\mathcal{CAT}$ of categories and functors, and - in Part 2 - we will do so. On the present stage the set theory that we use as a foundation guarantees that categories and functors are well-defined entities and allows for dealing with them in the obvious way.

The concept of natural transformation which we introduce now can be considered as an abstraction. Natural transformations are related to functors in a similar way as functors are related to categories.

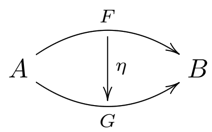
3.1 Definition. *Let* $A \overset{F}{\underset{G}{\rightrightarrows}} B$ *be functors. A natural transformation* $\eta\colon F \longrightarrow G$ *is a function* $\eta\colon \mathcal{O}_A \longrightarrow \mathcal{M}_B$ *such that*

- η_a *is a morphism* $Fa \longrightarrow Ga$ *in* B $\forall a \in \mathcal{O}_A$*;*
- *If* $x \xrightarrow{f} y$ *is a morphism in* A *then the diagram*

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

  is commutative.

We can visualize a natural transformation $\eta\colon F \longrightarrow G$ *by a diagram*



η *is called a natural isomorphism iff* η_a *is an isomorphism* $\forall a \in \mathcal{O}_A$*. In this case we write* η^{-1} *for the inverse natural isomorphism* $G \longrightarrow F$

$$(\eta^{-1})_a = {\eta_a}^{-1}.$$

We write $F \cong G$ *in case there exists a natural isomorphism* $\eta\colon F \longrightarrow G$*.*

In the sequel we will omit the word 'natural' from the notation. From now on **'transformation'** always means '**natural transformation**'.

3.2 Example (Bidual of a module)**.**
As in Example 2.11, consider the bidual functor

$$\star\star = \mathrm{Hom}_\Lambda(\bullet, \Lambda) \circ {}_\Lambda\mathrm{Hom}(\bullet, \Lambda)\colon {}_\Lambda\mathcal{MOD} \longrightarrow {}_\Lambda\mathcal{MOD}.$$

There is a well-known transformation

$$\eta\colon 1_{{}_\Lambda\mathcal{MOD}} \longrightarrow \star\star \qquad \eta_a(x)(\omega) = \omega(x)$$

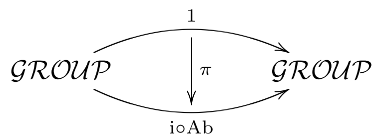
and similarly for right modules. If Λ *is a field, and* $\star\star$ *is restricted to the full subcategory of reflexive vector spaces, we obtain an isomorphism. This statement is equally valid for the categories of normed linear spaces over* $\mathbb{R}$ *or* $\mathbb{C}$*, and for the category of locally compact abelian groups.*

3.3 Example (Abelianization)**.**

Consider the composition

$$\mathcal{GROUP} \xrightarrow{\mathrm{Ab}} \mathcal{AB} \overset{i}{\hookrightarrow} \mathcal{GROUP}$$

of the abelianization with the forgetful functor i. The canonical epimorphism $\pi_G \colon G \longrightarrow G/G'$ defines a transformation



This is the reason why π_G is called canonical.

3.4 Example. *Let $\Lambda \in \mathcal{RING}$ and $A \in {}_\Lambda\mathcal{MOD}$. The isomorphism $A \cong \Lambda \otimes_\Lambda A$ is natural. Thus it defines an isomorphism*

$$1_{{}_\Lambda\mathcal{MOD}} \cong \Lambda \otimes_\Lambda \bullet \colon {}_\Lambda\mathcal{MOD} \longrightarrow {}_\Lambda\mathcal{MOD}.$$

3.5 Example (Adjoint Associativity)**.**

Let R, S, T denote rings with units. Consider three bimodules

$$A \in {}_R\mathcal{MOD}_S,\ B \in {}_S\mathcal{MOD}_T,\ C \in {}_R\mathcal{MOD}_T.$$

The association

$$\phi(f)(a)(b) = f(a \otimes b)$$

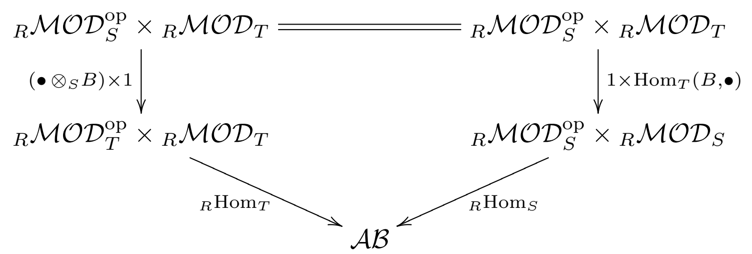
defines an isomorphism of abelian groups

$${}_R\mathrm{Hom}_T(A \otimes_S B, C) \cong {}_R\mathrm{Hom}_S(A, \mathrm{Hom}_T(B, C)).$$

For fixed B consider the associated functors

$$\bullet \otimes_S B : {}_R\mathcal{MOD}_S \longrightarrow {}_R\mathcal{MOD}_T \quad \text{and} \quad \mathrm{Hom}_T(B, \bullet) : {}_R\mathcal{MOD}_T \longrightarrow {}_R\mathcal{MOD}_S.$$

Combining functors according to the diagram



we obtain

$${}_R\mathrm{Hom}_T(\bullet \otimes_S B, \bullet) \quad \textit{and} \quad {}_R\mathrm{Hom}_S(\bullet, \mathrm{Hom}_T(B, \bullet))$$

which are isomorphic as functors

$${}_R\mathcal{MOD}_S^{\mathrm{op}} \times {}_R\mathcal{MOD}_T \longrightarrow \mathcal{AB}$$

by the above association. Therefore ϕ is a (natural) isomorphism.

3.6 Definition. *Given three functors* $F, G, H\colon A \longrightarrow B$ *and transformations* $F \xrightarrow{\eta} G \xrightarrow{\varepsilon} H$*, their* ***composition*** *is defined as*

$$(\varepsilon \circ \eta)_a = \varepsilon_a \circ \eta_a.$$

It is plain that ε is a transformation $\varepsilon \circ \eta\colon F \longrightarrow H$.

In case that A is a small category, this definition provides the category of A-diagrams in B.

3.7 Definition. *Let* A *be small. The category* B^A *consists of all functors* $A \longrightarrow B$ *as objects. Morphism between functors* $A \overset{F}{\underset{G}{\rightrightarrows}} B$ *are transformations* $F \longrightarrow G$*. The composition of* $F \xrightarrow{\eta} G$ *and* $G \xrightarrow{\varepsilon} H$ *in* B^A *is the composition* $\varepsilon \circ \eta$ *of transformations.*

It is convenient to go one step further and make transformations to morphisms without restricting them to fixed categories A, B. To this end one has to construct a product of transformations.

3.8 Definition. *Let* A, B, C *be arbitrary categories. Consider functors* $A \overset{F}{\underset{G}{\rightrightarrows}} B$ *and* $B \overset{H}{\underset{K}{\rightrightarrows}} C$*, and let* $\eta\colon F \longrightarrow G$ *and* $\delta\colon H \longrightarrow K$ *be transformations. For arbitrary objects* $a \in A$*, applying the naturality condition of* δ *to the morphism* η_a *results in a commutative square:*

$$\begin{array}{ccc} HF(a) & \xrightarrow{H(\eta_a)} & HG(a) \\ \downarrow{\scriptstyle \delta_{Fa}} & & \downarrow{\scriptstyle \delta_{Ga}} \\ KF(a) & \xrightarrow[K(\eta_a)]{} & KG(a) \end{array}$$

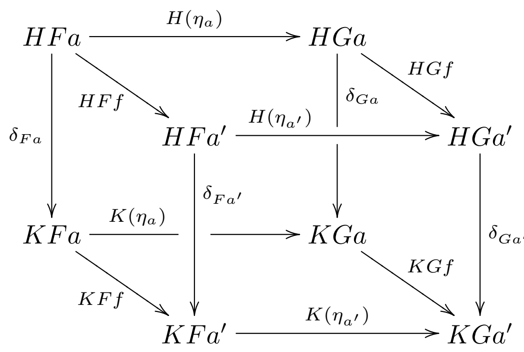
The **star product** *of* δ *and* η *is the family of diagonals of these squares*

$$\delta \star \eta\colon HF \longrightarrow KG, \quad (\delta \star \eta)_a = \delta_{Ga} \circ H(\eta_a) = K(\eta_a) \circ \delta_{Fa}.$$

Given a morphism $a \xrightarrow{f} a'$ in A, we get the commutative square

$$\begin{array}{ccc} Fa & \xrightarrow{\eta_a} & Ga \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fa' & \xrightarrow{\eta_{a'}} & Ga' \end{array}$$

Application of H and K to this square yield commutativity of the front and back square of the cube

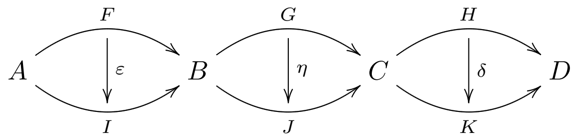


The remaining faces are commutative by the naturality conditions of δ with respect to the morphisms Ff, Gf, η_a and $\eta_{a'}$. Plainly then

$$\begin{aligned} KGf \circ (\delta \star \eta)_a &= KGf \circ K(\eta_a) \circ \delta_{Fa} = K(\eta_{a'}) \circ KFf \circ \delta_{Fa} = \\ &= K(\eta_{a'}) \circ \delta_{Fa'} \circ HFf = (\delta \star \eta)_{a'} \circ HFf. \end{aligned}$$

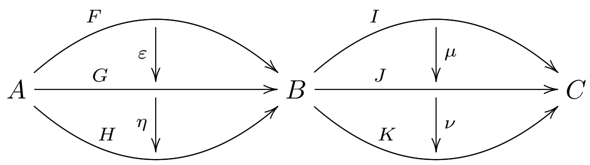
So the star product is indeed a transformation $\delta \star \eta \colon HF \longrightarrow KG$.

It is **associative**



$$(\delta \star \eta) \star \varepsilon = \delta \star (\eta \star \varepsilon)$$

and satisfies the **Interchange Law**



$$(\nu \circ \mu) \star (\eta \circ \varepsilon) = (\nu \star \eta) \circ (\mu \star \varepsilon).$$

3.9 Definition. *With* **nat** *we denote the category of small transformations. Its objects are arbitrary small categories. A morphism between small categories A, B is a triple (F, η, G), where $F, G \colon A \longrightarrow B$ are functors and $\eta \colon F \longrightarrow G$ is a transformation. The composition of*

$$A \xrightarrow{(F,\eta,G)} B \xrightarrow{(H\delta,K)} C$$

is defined by the star product

$$(H, \delta, K) \circ (F, \eta, G) = \quad A \xrightarrow{(HF,\delta\star\eta,KG)} C.$$

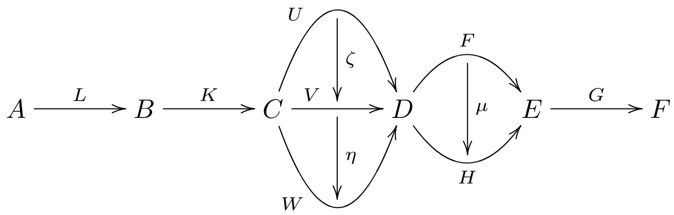
The Associativity Rule together with the Interchange Law ensure that **nat** is a category.

We introduce the convention that, in contexts involving the star product, we write F instead of 1_F. With this notation we have for example

$$(\mu \star F)_a = \mu_{Fa} \quad \text{and} \quad (K \star \eta)_a = K(\eta_a), \quad \text{even} \quad K \circ H = K \star H.$$

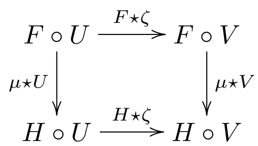
The following seven statements are the famous five rules of Godement.

3.10 Corollary (Godement's Five Rules)**.** *Given the following situation*



Then we have:

1. $(G \circ F) \star \zeta = G \star (F \star \zeta)$
2. $\zeta \star (K \circ L) = (\zeta \star K) \star L$
3. $1_U \star K = 1_{U \circ K}$
4. $F \star 1_U = 1_{F \circ U}$
5. $F \star (\zeta \star K) = (F \star \zeta) \star K$
6. $F \star (\eta \circ \zeta) \star K = (F \star \eta \star K) \circ (F \star \zeta \star K)$
7. *The square below is commutative:*



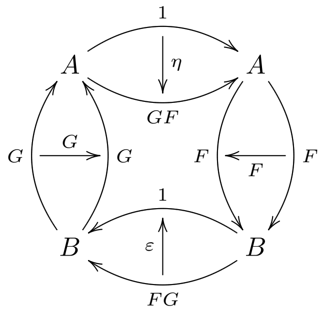
The notion of isomorphism between categories is a very rigid one. Interesting isomorphisms seldomly appear. There are several notions describing weaker and more useful concepts. The following defines a particularly important one.

3.11 Definition. *Let $F\colon A \longrightarrow B$ be a functor. F is an* **equivalence** *iff there exists a functor $G\colon B \longrightarrow A$ with $G \circ F \cong 1_A$ and $F \circ G \cong 1_B$. A and B are then called* **equivalent categories**, *written $A \sim B$.*

3.12 Theorem. *Let $F\colon A \longrightarrow B$ be a functor. The following are equivalent:*

1. *F is an equivalence.*
2. *There is a functor $G\colon B \longrightarrow A$ and natural isomorphisms $\eta\colon 1_A \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow 1_B$ such that $F \star \eta = (\varepsilon \star F)^{-1}$ and $G \star \varepsilon = (\eta \star G)^{-1}$.*

3. F is full, faithful and dense.



This is called an **equivalence situation** denotet by $(F, G, \eta, \varepsilon)$. Equivalences

combined by composition give an equivalence. The notion of equivalence is selfdual. If $(F, G, \eta, \varepsilon)$ is an equivalence situation, so is $(G, F, \varepsilon^{-1}, \eta^{-1})$.

Obviously isomorphic categories are equivalent. Equivalence is an equivalence relation on the class (conglomerate) of all categories which is weaker than the relation of being isomorphic.

3.13 Proposition. *Skeletal categories are equivalent if and only if they are isomorphic.*

Given a skeleton S of A, let $|a|$ be the unique object in S with $|a| \cong a \; \forall a \in A$. We can choose isomorphisms $\eta_a \colon a \cong |a|$ in such a way that $\eta_a = 1_a \forall a \in S$. Then

$$a \xrightarrow{f} b \;\mapsto\; |a| \xrightarrow{\eta_b f \eta_a^{-1}} |b|$$

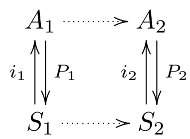
defines a functor $P \colon A \longrightarrow S$ with $S \overset{i}{\hookrightarrow} A \xrightarrow{P} S = 1_S$. So P is left inverse to the embedding i. Moreover $\eta \colon 1_A \cong iP$.

3.14 Theorem. *Categories are equivalent if and only if they have isomorphic skeletons.*

Proof. Consider the embeddings of skeletons and their retractions

$$i_k \colon S_k \longrightarrow A_k, \quad P_k \colon A_k \longrightarrow S_k \qquad (k = 1, 2)$$

and assume that $\Phi \colon S_1 \cong S_2$. Set $F := i_2 \Phi P_1 \colon A_1 \longrightarrow A_2$, and $G := i_1 \Phi^{-1} P_2 \colon A_2 \longrightarrow A_1$. Then $GF = i_1 P_1 \cong 1_{A_1}$. In the same way $FG \cong 1_{A_2}$, hence $A_1 \sim A_2$. Conversely assume that there is an equivalence $F \colon A_1 \longrightarrow A_2$. Choose a skeleton $S_1 \longrightarrow A_1$. Define S_2 as the full subcategory of A_2 whose object class is $\{F i_1(s) \mid s \in S_1\}$. This is a skeleton of A_2, and $P_2 F i_1 \colon S_1 \cong S_2$.



□

3.15 Definition. *A **duality** between categories A and B is an equivalence*

$$A^{\mathrm{op}} \sim B.$$

*A and B are then called **dually equivalent**.*

3.16 Example.

1. *Isomorphisms are equivalences; any category is equivalent to each of its skeletons.*

2. *Consider a functor $F\colon A \longrightarrow A$. If $F \cong 1_A$ then F is an equivalence.*

3. *For any field k, the category of finite dimensional vector spaces over k is equivalent to the category $\mathcal{MAT}_k$ of k-matrices.*

4. *Again, consider the category A of finite dimensional k-spaces. A well-known exercise from linear algebra is to show that the double dual of a finite dimensional space a is (naturally) isomorphic to a. Thus, the composition of the contravariant hom-functor $\star = \mathrm{Hom}_k(\bullet, k)\colon A^{\mathrm{op}} \longrightarrow A$ with itself*

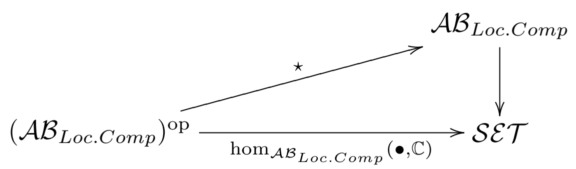
$$A \xrightarrow{\star} A^{\mathrm{op}} \xrightarrow{\star} A$$

is isomorphic to the identity functor on A. Consequently $\star$ is an equivalence of A^{op} with A, i.e., a duality of A. This equivalence is not an isomorphism (e.g., $\star$ is not bijective on objects).

5. *Consider the 1-dimensional torus $\mathbb{S}^1$ as an object of $\mathcal{AB}_{Loc.Comp}$, the locally compact abelian Hausdorff groups. If $a \in \mathcal{AB}_{Loc.Comp}$ is an arbitrary object, then the hom-set*

$$\hom_{\mathcal{AB}_{Loc.Comp}}(a, \mathbb{S}^1) \subseteq C(a, \mathbb{S})$$

is an abelian group with pointwise operations. The topolgy inherited from the compact-open topology on $C(a, \mathbb{S})$ turns it into an object of $\mathcal{AB}_{Loc.Comp}$. Thus, the contravariant hom-functor to the object $\mathbb{S}^1$ factors through $\mathcal{AB}_{Loc.Comp}$



*The group $a^\star$ is called the **dual group** of a, its elements are known as **characters**. As with the previous example, the composition of the character functor $\star$ with itself*

$$\mathcal{AB}_{Loc.Comp} \xrightarrow{\star} (\mathcal{AB}_{Loc.Comp})^{\mathrm{op}} \xrightarrow{\star} \mathcal{AB}_{Loc.Comp}$$

is isomorphic to the identity functor on $\mathcal{AB}_{Loc.Comp}$. This by associating

$$\eta_a\colon a \longrightarrow a^{\star\star}, \quad \eta_a(x)(\alpha) = \alpha(x).$$

*Consequently $\star$ is a duality of $\mathcal{AB}_{Loc.Comp}$, called the **Pontrjagin duality**.*

6. *Compact Hausdorff spaces are dually equivalent to $C^\star$-algebras by the* ***Gelfand-Naimark duality***

$$\star\colon \mathcal{TOP}_{Comp.T2} \longrightarrow C^\star - \mathcal{ALG}, \quad X \mapsto C(X, \mathbb{C}).$$

7. *Consider the categories $\mathcal{BS}$ and $\mathcal{BA}$ of Boolean spaces and Boolean algebras respectively (cf. Example 2.4). The functors*

$$\begin{array}{l} \mathcal{BS}^{\mathrm{op}} \hookrightarrow \mathcal{TOP}^{\mathrm{op}} \xrightarrow{\chi} \mathcal{BA} \quad \textit{and} \\ \mathcal{BA}^{\mathrm{op}} \xrightarrow{S} \mathcal{BS} \end{array}$$

provide an equivalence. This is the ***Stone duality****.*

8. *Consider the functor $\mathbf{k}[\bullet]\colon \mathcal{AFF}(\mathbf{K}/\mathbf{k})^{\mathrm{op}} \longrightarrow \mathcal{ALG}_{\mathbf{k}}$, (cf. Example 2.4/4). If $\mathbf{K}$ is algebraically closed, then $\mathbf{k}[\bullet]$ is an equivalence of $\mathcal{AFF}(\mathbf{K}/\mathbf{k})^{\mathrm{op}}$ with the full subcategory of $\mathcal{ALG}_{\mathbf{k}}$ consisting of finitely generated, reduced $\mathbf{k}$-algebras.*

9. *An* ***affine scheme*** *is a locally ringed space which is isomorphic (in the category of locally ringed spaces) to the spectrum of a commutative ring. The association $R \mapsto \operatorname{Spec} R$ defines a functor*

$$\operatorname{Spec}\colon \mathcal{CRING}^{\mathrm{op}} \longrightarrow \textit{affine schemes}.$$

The map associating to any ringed space its global sections

$$(X, \mathcal{O}) \mapsto \Gamma(X, \mathcal{O})$$

defines a contravariant functor

$$\Gamma\colon \textit{affine schemes}^{\mathrm{op}} \longrightarrow \mathcal{CRING}.$$

Spec *together with* Γ *provide a duality*

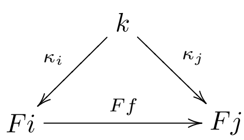
$$\mathcal{CRING}^{\mathrm{op}} \sim \textit{affine schemes}.$$

This list of examples of equivalences could be extended indefinitely.

3.2 Limits and Colimits

3.17 Definition. *Let I be a small category and $F\colon I \longrightarrow A$ a functor.*[29]

1. *A* **natural source** *for F, also called a* **cone** *over F, is a pair $(k, (\kappa_i)_{i \in \mathcal{O}_I})$ with $k \in \mathcal{O}_A$ and $\kappa_i\colon k \longrightarrow Fi$ a morphism in A $\forall i$, such that, for each morphism $i \xrightarrow{f} j$ in I, the triangle*

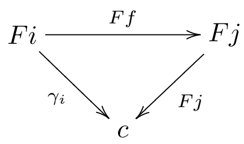


commutes.[30] *The category* Cone_F *of cones over F is the following:*

[29] A functor $I \longrightarrow A$ with I a small category is also called a **diagram**.
[30] Again we shall skip the adjective 'natural'.

- $\mathcal{O}_{\mathrm{Cone}_F}$ *is the class of all sources over* F.
- *A morphism* $(k, (\kappa_i)) \longrightarrow (l, (\lambda_i))$ *is a morphism* $h\colon k \longrightarrow l$ *in* A *such that* $\lambda_i h = \kappa_i \ \forall i \in I$.
- *Composition in* Cone_F *is composition in* A.

2. *Dually, a* ***sink*** *for* F *- also called a* ***cone below*** F*, or a* ***cocone of*** F *- is a pair* $((\gamma_i)_{i\in\mathcal{O}_I}, c)$ *with* $c \in \mathcal{O}_A$, $\gamma_i\colon Fi \longrightarrow c$ $(i \in I)$*, such that, for any* I*-morphism* $i \xrightarrow{f} i'$, $\gamma_{i'} \circ Ff = \gamma_i$*, i.e., the triangle*



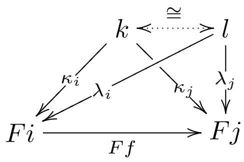
commutes. Cone^F *is the category of cones below* F.

3. *A terminal object of* Cone_F *is a* ***limit*** *of* F.

4. *Dually, an initial object of* Cone^F *is a* ***colimit*** *of* F.

3.18 Proposition. *Let* $(k, (\kappa_i))$ *and* $(l, (\lambda_i))$ *be two limits of the diagram* $F\colon I \longrightarrow A$*. Then there is an isomorphism* $\varphi\colon k \longrightarrow l$ *in* A *unique with respect to the property* $\lambda_i\varphi = \kappa_i \ \forall i$.

$i \xrightarrow{f} j \Longrightarrow$



Likewise any two colimits of F *are isomorphic by a unique isomorphism compatible with the cone-morphisms.*

Proof. Any two initial (terminal) objects are isomorphic via a unique isomorphism. □

This allows us to talk about **the limit** of F.

3.19 Definition. *Let* $I \xrightarrow{f} A$ *be a diagram. We write*

$$\begin{aligned} \lim_{\leftarrow} F = \lim_{\leftarrow} Fi &= \textit{limit of } F \\ \lim_{\rightarrow} F = \lim_{\rightarrow} Fi &= \textit{colimit of } F \end{aligned}$$

in case such entities do exist.

The equality sign in Definition 3.19 is usual, despite the fact that a limit is in general not unique. There is but sufficient uniqueness guaranteed by Proposition 3.18. Note that a limit is a pair $(k, (\kappa_i)_{i\in I})$. In concrete situations, when evident from context, the cone-morphism part is omitted. Sometimes we write $\operatorname{colim} F$ to denote the colimit of F.

Sources can be considered by using transformations from constant functors.

A cone $(k, (\kappa_i)_{i \in \mathcal{O}_i})$ over $F \colon I \longrightarrow A$ is a transformation

$$\kappa \colon C_k \longrightarrow F.$$

Similarly, a cocone $((\gamma_i)_{i \in \mathcal{O}_i}, c)$ below $F \colon I \longrightarrow A$ is a transformation

$$\gamma \colon F \longrightarrow C_c.$$

The terminlogy is not uniform. What we call 'limit' is often called **projective limit**. A 'colimit' is also called **inductive limit**.

By varying the domain of the functor F, limits and colimits provide a variety of prominent constructions.

3.20 Definition. *Let $F \colon I \longrightarrow A$ be a diagram.*

1. *If I is discrete, then F is just a map from the set I to the object class of A. The limit of F is called* **product**

$$\varprojlim F = \left(\prod_{i \in I} F(i), (\pi_i)_{i \in I} \right).$$

The colimit of F is the ***coproduct***

$$\varinjlim F = \left(\coprod_{i \in I} F(i), (\iota_i)_{i \in I} \right).$$

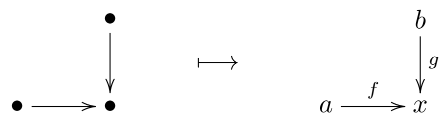
Note that a limit consists of an object and a family of morphisms. Often these morphisms are suppressed from notation. If I is a finite set $\{i_1, \dots, i_n\}$ we write as usual

$$F(i_1) \times \cdots \times F(i_n)$$

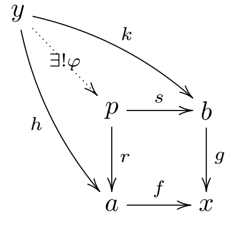
for the product. The coproduct is then written as

$$F(i_1) \sqcup \cdots \sqcup F(i_n).$$

2. *Consider two arrows with equal codomain $a \xrightarrow{f} x \xleftarrow{g} b$ in A. The* ***pullback*** *of f, g is the limit of the diagram*



This amounts to a pair of arrows with common domain $a \xleftarrow{r} p \xrightarrow{s} b$ such that $fr = gs$, and so that for arbitrary $a \xleftarrow{h} y \xrightarrow{k} b$ with $fh = gk$ there is a unique $\varphi \colon y \longrightarrow p$ such that $r\varphi = h$ and $s\varphi = k$.



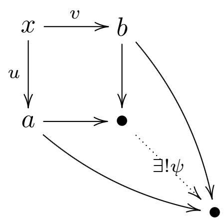
(2)

Diagram (2) is called a **pullback square***, r is a pullback of g along f, s a pullback of f along g.*[31]

In case f is a monomorphism, s is called an **inverse image** *of f along g.*

A pullback (2) *with $f = g$ is a* **congruence relation** *of f, also called a* **kernel pair**. *This is a minimal pair (r, s) such that* $\mathrm{dom}(r) = \mathrm{dom}(s)$, $\mathrm{cod}(r) = \mathrm{cod}(s) = \mathrm{dom}(f)$ *and* $fr = fs$.

The dual concept, called ***pushout****, is the colimit of a diagram with domain* $\bullet \longleftarrow \bullet \longrightarrow \bullet$. *A full square involving a pushout is called a* **pushout square**.



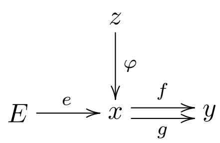
3. *Consider a pair* $x \underset{g}{\overset{f}{\rightrightarrows}} y$ *of parallel arrows in A. The* ***equalizer*** $\mathrm{Eq}(f, g)$ *of f and g is the limit of the functor*

$$\bullet \rightrightarrows \bullet \quad \longmapsto \quad x \underset{g}{\overset{f}{\rightrightarrows}} y$$

This means

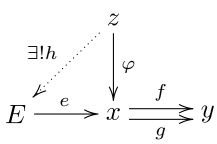
$$\mathrm{Eq}(f, g) = \lim_{\leftarrow}(\ x \underset{g}{\overset{f}{\rightrightarrows}} y\ ) = (E, e)$$

where $E \xrightarrow{e} x$ with $fe = ge$ and every diagram



with $f\varphi = g\varphi$

can be extended with a unique $h\colon z \longrightarrow k$ to a commutative diagram



[31]We will express this situation and similar ones by saying that (p, r, s) - or (r, s) alone - is **minimal** with respect to the condition $fr = gs$.

Dually, the ***coequalizer*** *of f, g is the colimit*

$$\operatorname{coEq}(f,g) = \lim_{\rightarrow} \left(x \overset{f}{\underset{g}{\rightrightarrows}} y \right).$$

As equalizers (coequalizers) are certain limits they are unique up to isomorphism. We say that a category A ***has equalizers (has coequalizers)*** *provided that every pair of parallel arrows in A has an equalizer (a coequalizer).*

4. *Let A be a pointed category, and $x \xrightarrow{f} y$ in A. The* ***kernel*** *of f is the equalizer of f and 0*

$$\ker f = \operatorname{Eq}(f,0) = \lim_{\leftarrow} \left(x \overset{f}{\underset{0}{\rightrightarrows}} y \right) = (K,k),$$

where $k \xrightarrow{k} x \xrightarrow{f} y = 0$ and every diagram

$$\begin{array}{ccccc} & & z & & \\ & & \downarrow{\scriptstyle g} & & \\ K & \xrightarrow{k} & x & \xrightarrow{f} & y \end{array} \qquad \textit{with } fg = 0$$

can be extended with a unique $h\colon z \longrightarrow k$ to a commutative diagram

$$\begin{array}{ccccc} & & z & & \\ & \swarrow{\scriptstyle \exists! h} & \downarrow{\scriptstyle g} & & \\ K & \xrightarrow{k} & x & \xrightarrow{f} & y \end{array}$$

Dually, the ***cokernel*** *of f is the colimit*

$$\operatorname{coker} f = \operatorname{coEq}(f,0) = \lim_{\rightarrow} \left(x \overset{f}{\underset{0}{\rightrightarrows}} y \right).$$

Again, kernels (cokernels) are unique up to isomorphism.

3.21 Definition. *A pointed category A is said to* ***have kernels (cokernels)****, provided that each of its morphism has a kernel (a cokernel).*

By its definition, $\ker f$ consists of an object part and a morphism part. The latter determines the object part as its domain, so it carries the entire information. The same is true for cokernels. Therefore, e.g., the construct 'coker($\ker f$)' does make sense. In the category of groups, for example, the kernel

$$k \xrightarrow{\eta} G = \ker(G \xrightarrow{f} H)$$

is completely described by the subgroup $\eta(k) \subseteq G$. This is the reason why in group theory and similar algebraic theories a kernel is considered as a subgroup/subalgebra rather than a morphism. A similar remark applies to cokernels. We will allow for lazy notation, writing $\ker f$ and $\operatorname{coker} f$ for the object or the morphism part when convenient. The same convention will be used for arbitrary limits.

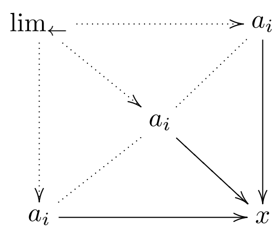
The concepts 'pullback' and 'equalizer' extend to larger domains.

3.22 Definition.

1. *Given a set-indexed family of morphisms with common codomain*

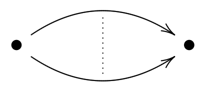
$$a_i \xrightarrow{f_i} x \quad (i \in I).$$

The limit of the corresponding diagram is called a ***multiple pullback****.*



Dually, a family $y \xrightarrow{g_i} b_i$ *produces a* ***multiple pushout****.*

2. *A* ***multiple equalizer (coequalizer)*** *is the limit (colimit) of a diagram*



3.23 Definition. *Let I be a quasiordered set, $F\colon I \longrightarrow \mathcal{SET}$ a functor. Then the limit of F is $(L, (\lambda_i)_{i\in I})$, where*

$$L = \{x \in \prod_{i\in I} F(i) \mid F_{ij}(x_i) = x_j \ \forall i, j \text{ with } i \leq j\}$$

and $\lambda_i\colon L \longrightarrow F(i)$ be the restrictions of the projections $\prod_{i\in I} F(i) \longrightarrow F(i)$.

3.24 Definition. *Now let $(Q, \leq)$ be a quasiordered set with the aditional property $\forall i, j \in Q \exists k \in Q$ such that $i \leq k \wedge j \leq k$. A functor $D\colon Q \longrightarrow \mathcal{A}$ is called a* `direct system` *in a category $\mathcal{A}$. A colimit of F is called* `direct limit`*. If F has values in $\mathcal{SET}$ it can be constructed in the following way.*

$$\varinjlim D = \mathrm{colim}_{i\in Q} D(i).$$

3.25 Example. *For any topological space X let $\hat{X}$ be the topology*[32] *of X equipped with the relation $U \leq V \leftrightarrow U \supseteq V$. Then $\hat{X}$ is a direct sustem. For $p \in X$ the set of open neighbourhoods of p, $\mathcal{U}^0(p)$ is a direct subsystem of $\hat{X}$. We will always consider topologies and neighbourhood systems this way.*

3.26 Theorem (Colimit Construction)**.** *Let $\mathcal{A}$ be a concrete 'algebraic' category, $D\colon (Q, \leq) \longrightarrow \mathcal{A}$ a direct system. Then the direct limit $\mathrm{colim}_{i\in Q} D(i)$ exists. Moreover it can be constructed from the corresponding set-functor.*

[32] i.e. the set of all open sets of X.

Proof. Consider the equivalence on $\bigcup_{i\in Q}\{i\}\times|D(i)|$ given by

$$(i,x)\sim(j,y) \iff \exists k\in Q \text{ with } i\le k\wedge j\le k \text{ and } D_{ik}(x)=D_{jk}(y).$$

Set $C:=\bigcup_{i\in Q}\{i\}\times|D(i)|/\sim$, $\kappa_i\colon D(i)\longrightarrow C,\ x\mapsto[(i,x)]$.

If $i\le j, x\in D(i)$ and $y=D_{ij}(x)$ then $(i,x)\sim(j,y)$. This shows $\kappa_j\circ D_{ij}=\kappa_i$, i.e., $(C,(\kappa_i)_{i\in Q})$ is a co-cone below D. Let $(A,(\varphi_i)_{i\in Q})$ be any co-cone. Take an $\alpha\in C$, there are then i,x with $\alpha=[(i,x)]$. If $[(i,x)]=[(j,y)]$ then $D_{ik}(x)=D_{j,k}(y)$ for some $k\ge i,j$ hence $\varphi_i(x)=\varphi_k D_{ik}(x)=\varphi_k D_{jk}(y)=\varphi_j(y)$. This shows that the association $[(i,x)]\mapsto\varphi_i(x)$ defines a map $\Phi\colon C\longrightarrow A$, and $\Phi\circ\kappa_i=\varphi_i$. Obviously Φ is unique, and so $(C,(\kappa_i)_{i\in Q})$ is a colimit of D as a set functor. If $\mathcal{A}=\mathcal{SET}$ we are done.

Now assume that the objects of $\mathcal{A}$ are sets equipped with some algebraic structure which is preserved by morphisms and let f_μ denote an $n-$ary operation symbol in $\mathcal{A}$. If $n=0$ then the constants $f_\mu^i(0)\in D(i)$ $(i\in Q)$ get all identified in C defining the unique element $f_\mu(0)=[i,f_\mu^i(0)]$.

For $n>0$ take $\alpha_\nu=[(i_\nu,x_\nu)]$ in C, $1\le\nu\le n$. Choose $k\in Q$ with $i_\nu\le k\ \forall\nu$ and define

$$f_\mu(\alpha_1,\ldots,\alpha_n)=\kappa_k\left(f_\mu(D_{i_1,k}(x_1),\ldots,D_{i_n,k}(x_n))\right).$$

If (j_ν,y_ν) are other representants of α_ν and $l\ge j_\nu$ then there are m_ν with $i_\nu,j_\nu\le m_\nu$ such that $D_{i_\nu,m_\nu}(x_\nu)=D_{j_\nu,m_\nu}(y_\nu)$. Choose $s\in Q$ with $s\ge m_1,\ldots,m_n,k,l$; then

$$\begin{aligned}
D_{ks}\left(f_\mu(D_{i_1,k}(x_1),\ldots,D_{i_n,k}(x_n))\right) &= f_\mu(D_{ks}D_{i_1,k}(x_1),\ldots,D_{ks}D_{i_n,k}(x_n))\\
&= f_\mu(D_{i_1,s}(x_1),\ldots,D_{i_n,s}(x_n)) \quad \text{while}\\
D_{ls}\left(f_\mu(D_{j_1,l}(y_1),\ldots,D_{j_n,l}(y_n))\right) &= f_\mu(D_{ls}D_{j_1,l}(y_1),\ldots,D_{ls}D_{j_nl}(y_n))\\
&= f_\mu(D_{j_1,s}(y_1),\ldots,D_{j_n,s}(y_n)).
\end{aligned}$$

But $D_{i_\nu,s}(x_\nu)=D_{m_\nu,s}D_{i_\nu,m_\nu}(x_\nu)=D_{m_\nu,s}D_{j_\nu,m_\nu}(y_\nu)=D_{j_\nu,s}(y_\nu)$ hence $(k,f_\mu(D_{i_1,k}(x_1),\ldots,D_{i_n,k}(x_n)))\sim(l,f_\mu(D_{j_1,l}(y_1),\ldots,D_{j_n,l}(y_n)))$, that means, $f_\mu(\alpha_1,\ldots,\alpha_n)$ is well defined.

One immediately realizes that algebraic axioms stay valid in C if they hold in the category $\mathcal{A}$.

For elements $x_1,\ldots,x_n\in D(i)$ we see that $f_\mu(\kappa_i(x_1),\ldots,\kappa_i(x_n))=[i,f_\mu(x_1,\ldots,x_n)]$, whence $\kappa_i\colon D(i)\longrightarrow C$ is a morphism in $\mathcal{A}$.

Again consider an arbitrary co-cone $(A,(\varphi_i)_{i\in Q})$ below D in $\mathcal{A}$. The uniquely determined map $\Phi\colon C\longrightarrow A$ is then a morphism in $\mathcal{A}$:

$\alpha_1=[(i_1,x_1)],\ldots,\alpha_n=[(i_n,x_n)]$, so $f_\mu(\alpha_1,\ldots,\alpha_n)=[(j,f_\mu(D_{i_1,j}(x_1),\ldots,D_{i_n,j}(x_n)))]$ where $j\ge i_1,\ldots,i_n$. Then

$$\begin{aligned}
&\Phi(f_\mu(\alpha_1,\ldots,\alpha_n))=\varphi_j(f_\mu(D_{i_1,j}(x_1),\ldots,D_{i_n,j}(x_n)))\\
=\ &f_\mu(\varphi_jD_{i_1,j}(x_1),\ldots,\varphi_jD_{i_n,j}(x_n))=f_\mu(\varphi_{i_1}(x_1),\ldots,\varphi_{i_n}(x_n))\\
=\ &f_\mu(\Phi[(i_1,x_1)],\ldots,\Phi[(i_n,x_n)])=f_\mu(\Phi(\alpha_1),\ldots,\Phi(\alpha_n))
\end{aligned}$$

This shows that $(C,(\kappa_i)_{i\in Q})$ is a colimit of D. □

4 Sheaves

4.1 Definition. *Let X be a space, $\mathcal{A}$ a concrete category.*

1. *A* **presheaf** *on X (with values in $\mathcal{A}$) is a functor $F\colon \hat{X} \longrightarrow \mathcal{A}$. For $x \in F(U)$ and $V \subseteq U$, we write $x|V$ for the image of x under the morphism $F_{UV}\colon F(U) \longrightarrow F(V)$. Elements of $F(U)$ are called* **sections** *over U, sometimes the notation $\Gamma(U,F)$ is used to denote $F(U)$. The elements of $\Gamma(X,F)$ are called* **global sections**.

2. *A* **sheaf** *is a presheaf which fulfills the following additional* sheaf axiom*:*

   - *If $U = \bigcup_\alpha V_\alpha$ is an open covering of $U \in \hat{X}$, $x_\alpha \in F(V_\alpha)$ such that $x_\alpha|V_\alpha \cap V_\beta = x_\beta|V_\alpha \cap V_\beta$ $\forall \alpha,\beta$ then $\exists! x \in F(U)$ with $x|V_\alpha = x_\alpha$ $\forall\alpha$.*

The sheaf axiom has as immediate consequences:

1. If $U = \bigcup_\alpha V_\alpha$ is an open covering of $U \in \hat{X}$, $x,y \in F(U)$ such that $x|V_\alpha = y|V_\alpha$ $\forall\alpha$ then $x = y$.

2. $\operatorname{card}(|F(\emptyset)|) = 1$.

We consider presheaves as objects of the functor category $\mathcal{A}^{\hat{X}}$ and, consequently, a **morphism of (pre)sheaves** $\varphi\colon F \longrightarrow G$ is just a natural transformation $F \longrightarrow G$.

4.2 Definition. *For any topological space X and any 'algebraic' category $\mathcal{A}$ we denote the full subcategory of $\mathcal{A}^{\hat{X}}$ of sheaves by $\mathcal{G} = \mathcal{G}(X,\mathcal{A})$*

4.3 Definition. *Let $F\colon \hat{X} \longrightarrow \mathcal{A}$ be a presheaf, $p \in X$ a point. The* **stalk** *at p is*

$$F_p = \operatorname{colim}_{U \in \mathcal{U}^\circ(p)} F(U).$$

For each $p \in X$ the stalk F_p exists since $F|\mathcal{U}^\circ(p)$ is a direct system.

4.4 Example.

1. *The product in the categories $\mathcal{SET}, \mathcal{SGRP}, \mathcal{MON}$, $\mathcal{CMON}$, $\mathcal{GROUP}$, $\mathcal{AB}$, $\mathcal{RING}$, $\mathcal{CRING}$, ${}_\Lambda\mathcal{MOD}, \mathcal{ALG}_k, \mathcal{CALG}_k$ etc. is the usual cartesian product equipped with the appropriate algebraic structure.*

2. *The product of a family $(X_i)_{i\in I}$ in $\mathcal{TOP}$ is the topological product. This is the product of the underlying sets, together with the initial topology with respect to projections $\pi_i\colon \prod_{i\in I} X_i \longrightarrow X_i$.*

3. *The product in $\mathcal{TOP}_n$ is*

$$\prod_{i\in I} X_i = \prod_{i\in I}(X_i^1,\ldots,X_i^n) = \Big(\prod_{i\in I} X_i^1,\ldots,\prod_{i\in I} X_i^n\Big),$$

   where $\prod_{i\in I} X_i^k$ is the product in $\mathcal{TOP}$, with projections

$$\pi_i^k\colon \prod_{i\in I} X_i^k \longrightarrow X_i^k,\ (i \in I,\ 1 \le k \le n).$$

4. *The product in $\mathcal{TOP}_\star$ is*

$$\prod_{i\in I}(X_i, x_0^i) = \Big(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\Big).$$

5. *The coproduct in $\mathcal{SET}$ and $\mathcal{TOP}$ is the disjoint union, and the disjoint union equipped with final topology respectively.*

6. *The coproduct in $\mathcal{TOP}_n$ is*

$$\coprod_{i\in I} X_i = \coprod_{i\in I}(X_i^1,\dots,X_i^n) = \Big(\coprod_{i\in I} X_i^1,\dots,\coprod_{i\in I} X_i^n\Big),$$

where $\coprod_{i\in I} X_i^k$ is the coproduct in $\mathcal{TOP}$, with injections

$$\iota_i\colon X_i^k \longrightarrow \coprod_{i\in I} X_i^k \ (i\in I,\ 1\le k\le n).$$

7. *The coproduct in $\mathcal{TOP}_\star$ is the **wedge sum**. This is the coproduct of the underlying spaces (the disjoint union), with all basis points identified*

$$\bigvee_{i\in I}(X_i,x_0^i) = \Big(\coprod_{i\in I} X_i/\{x_0^i \mid i\in I\},\langle x_0\rangle\Big)$$

where $\langle x_0\rangle$ - the class of individual basis points - is the basis point.

8. *The coproduct in ${}_\Lambda\mathcal{MOD}$ is the direct sum together with injections*

$$\mu_i\colon a_i \longrightarrow \bigoplus_{i\in I} a_i,\ x\mapsto \Big((0)_{j\in I}\Big)^{i/x} \quad (i\in I).$$

9. *Let k denote a commutative ring. The coproduct of two algebras $a,b\in\mathcal{ALG}_k$ is the tensor product*

$$a\sqcup b = (a\otimes_k b,\iota_1,\iota_2)$$

with injections

$$\iota_1\colon a\longrightarrow a\otimes_k b,\ x\mapsto x\otimes 1; \qquad \iota_2\colon b\longrightarrow a\otimes_k b,\ y\mapsto 1\otimes y.$$

10. *Let $k \xrightarrow{\sigma} a$ and $k \xrightarrow{\tau} b$ be morphisms in $\mathcal{CRING}$. a and b become k-algebras via*

$$c\cdot x = \sigma(c)x,\quad c\cdot y = \tau(c)y \qquad (c\in k, x\in a, y\in b).$$

It follows that

$$\begin{array}{ccccc} & & a & & \\ & \overset{\sigma}{\nearrow} & & \overset{\iota_1}{\searrow} & \\ k & & & & a\otimes_k b \\ & \underset{\tau}{\searrow} & & \underset{\iota_2}{\nearrow} & \\ & & b & & \end{array}$$

is a pushout square in $\mathcal{CRING}$.

11. *Let* $X \xrightarrow{(f,f^\sharp)} Z$ *and* $Y \xrightarrow{(g,g^\sharp)} Z$ *be morphisms of affine schemes. Then there are commutative rings* A, B, C *such that*

$$X \cong \operatorname{Spec} A, \quad Y \cong \operatorname{Spec} B, \quad Z \cong \operatorname{Spec} C.$$

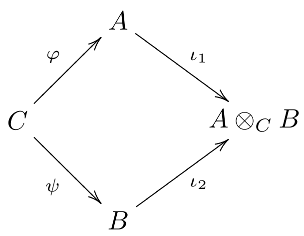
Since the spectrum

$$\operatorname{Spec}\colon \mathcal{CRING}^{\mathrm{op}} \longrightarrow \text{loc.ringed.spaces}$$

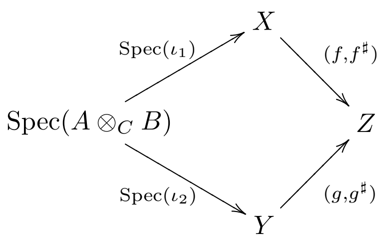
is a full functor, the morphisms $(f, f^\sharp)$ *and* $(g, g^\sharp)$ *come from ring homomorphisms*

$$(f, f^\sharp) = \operatorname{Spec}(\varphi) \quad \textit{and} \quad (g, g^\sharp) = \operatorname{Spec}(\psi)$$

where $\varphi\colon C \longrightarrow A$ *and* $\psi\colon C \longrightarrow B$*. Constructing the pushout in* $\mathcal{CRING}$



and taking the spectrum of this commutative diagram we obtain the pullback in the category of affine schemes



(3)

This construction is also called a ***fibered product****.*

12. *Let* A *have finite products,* $a \in A$ *an object. Assume chosen for each object* $x \in A$ *a product* $x \times a$*. Then the assignment* $x \mapsto x \times a$ *defines an endofunctor on* A*. For* $A = \mathcal{TOP}$ *and* $a = [0,1] \subset \mathbb{R}$*, this construction provides the* ***cylinder*** *over a space* x

$$\operatorname{Cyl}\colon \mathcal{TOP} \longrightarrow \mathcal{TOP}, \ x \mapsto x \times [0,1].$$

13. *Consider a category* I *with initial object* i_0*. Let* $d_i\colon i_0 \longrightarrow i$ *denote the unique morphisms with domain* i_0*. Let* $F\colon I \longrightarrow A$ *be a functor, where* A *is an arbitrary category. Then*

$$\varprojlim F = \big(Fi_0, (Fd_i)_{i \in \mathcal{O}_I}\big).$$

4.5 Definition.

1. *Assume that the product* $(\prod_i a_i, (\pi_i)_i)$ *exists. For* $f_i\colon x \longrightarrow a_i$ $(i \in I)$*, the unique morphism* $h\colon x \longrightarrow \prod_i a_i$ *minimal with respect to* $\pi_i h = f_i$ $\forall i$ *is written* $\langle (f_i)_{i\in I}\rangle$.

2. *Dually, if* $(\coprod_i a_i, (\iota_i)_i)$ *exists, and* $g_i\colon a_i \longrightarrow y$ $\forall i$*, then* $[(g_i)_{i\in I}]$ *denotes the unique morphism* $k\colon \coprod_i a_i \longrightarrow y$ *minimal with respect to* $k\iota_i = g_i$.

If I is the finite set $\{1,\dots,n\}$, we write $\langle f_1,\dots f_n\rangle$ and $[f_1,\dots,f_n]$ respectively. Thus we have

$$< h\pi_1,\dots,h\pi_n >= h \quad \text{and} \quad [k\iota_1,\dots,k\iota_n] = k.$$

4.6 Definition. *Consider the constant functor* $C_a\colon I \longrightarrow A$*, where* I *is discrete.*

- *If* $\prod_{i\in I} C_a = \prod_{i\in I} a$ *exists, we write* Δ_a *for the* **diagonal map**

$$\Delta_a := \langle (1_a)_{i\in I}\rangle.$$

- *In case* $\coprod_{i\in I} C_a = \coprod_{i\in I} a$ *does exist, the* **codiagonal map** *is*

$$\nabla_a := [(1_a)_{i\in I}].$$

4.7 Definition. *Let* A, I *be categories.*

- A *is* I-**complete** $\iff \forall F\colon I \longrightarrow A\ \exists \lim F$.
- A *is* **complete** $\iff$ A *is* I*-complete for each small category* I.
- *Dual notions:* I-**cocomplete** *and* **cocomplete**.
- A *is called* ***complete (cocomplete)*** *iff* A *is* I*-complete (*I*-cocomplete) for every small category* I.

For those classes of domain categories I whose limits have specific names, I-completeness of A is expressed by saying that A **has** these limits. So we say that A has products, provided that A is I-complete for every small discrete category I. Similarly A has equalizers, pullbacks, pushouts, finite products, etc., when A is I-complete for the respective category I.

4.8 Proposition (Discrete Associativity)**.** *Assume* A *has products. Let* I *be a set and* $J\colon I \longrightarrow \mathbf{V}$ *a set-valued function. Let, for all* $i \in I$, $x_i\colon J_i \longrightarrow \mathcal{O}_A$ *be a family of* A*-objects. Write* K *for the set* $\bigcup_{i\in I}\{i\} \times J_i$*. If, for each* $i \in I$, $(p_i, (\pi_{ij})_{j\in J_i})$ *is a product of* $(x_{ij})_{j\in J_i}$*, and* $(R, (\rho_i)_{i\in I})$ *a product of the family* $(p_i)_{i\in I}$*, then* $(R, (\pi_{ij}\rho_i)_{(i,j)\in K})$ *is a product of* $x\colon K \longrightarrow \mathcal{O}_A$*. This is formulated conveniently as*

$$\prod_{i\in I}\prod_{j\in J_i} x_{ij} = \prod_{(i,j)\in K} x_{ij}.$$

Dually, if A *has coproducts, then*

$$\coprod_{i\in I}\coprod_{j\in J_i} x_{ij} = \coprod_{(i,j)\in K} x_{ij}.$$

4.9 Proposition (Pullback Construction)**.** *Let A be a category which has equalizers and finite products.*[33] *Given $x \xrightarrow{f} z$ and $y \xrightarrow{g} z$. Take a product $(x \times y, \pi_0, \pi_1)$ of x and y, and choose an equalizer $e \xrightarrow{\eta} x \times y = \mathrm{Eq}(f \circ \pi_0, g \circ \pi_1)$*

$$\begin{array}{ccccc} & & x & & \\ & & \uparrow{\scriptstyle \pi_0} & \searrow{\scriptstyle f} & \\ e & \xrightarrow{\eta} & x \times y & & z \\ & & \downarrow{\scriptstyle \pi_1} & \nearrow{\scriptstyle g} & \\ & & y & & \end{array}$$

Then $(e, \pi_0\eta, \pi_1\eta)$ is a pullback of $x \xrightarrow{f} z$, $y \xrightarrow{g} z$.

4.10 Proposition.

1. *Every pullback of a monomorphism is a monomorphism. Precisely: Let*

$$\begin{array}{ccc} p & \xrightarrow{f'} & \bullet \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow{f} & \bullet \end{array}$$

   be a pullback square. Then, f monic implies f' monic.

2. *Every pullback of a retraction is a retraction.*

3. *$a \xrightarrow{f} b$ is a monomorphism $\iff$ $(1_a, 1_a)$ is a congruence relation of f.*

Proof. Given arrows $x \overset{r}{\underset{s}{\rightrightarrows}} p$ such that $f'r = f's$. Then, because $g \circ f'r = f \circ g'r$, there is exactly one $\varphi\colon x \longrightarrow p$ with $f'\varphi = f'r$ and $g'\varphi = g'r$. From the assumption we derive

$$fg'r = gf'r = gf's = fg's$$

and, because f is monic, we obtain $g'r = g's$. This means that $r = \varphi = s$. □

Of course we also have: g monic $\Longrightarrow$ g' monic.

4.11 Proposition. *Equalizers are monomorphisms. Dually, coequalizers are epimorphisms. Consequently, kernels are monic, cokernels are epi.*

Proof. Assume $(E, e) = \mathrm{Eq}(f, g)$ and let h, k be morphisms such that $eh = ek$.

$$z \overset{h}{\underset{k}{\rightrightarrows}} e \xrightarrow{e} x \overset{f}{\underset{g}{\rightrightarrows}} y$$

Then $feh = geh$, thus there is a unique $\varphi\colon Z \longrightarrow E$ with $e\varphi = eh$. Consequently $h = k$. □

4.12 Definition. *Consider a morphism $h\colon a \longrightarrow b$.*

[33] By the last remark, this is a short form for the phrase: 'A is I-complete for the categories $I = \bullet \rightrightarrows \bullet$ and $I = \{0, 1\}$'.

- *h is a **regular monomorphism** $\iff$ $\exists f, g$ such that $h = Eq(f,g)$.*
- *h is a **regular epimorphism** $\iff$ $\exists f, g$ such that $h = coEq(f,g)$.*
- *h is a **normal monomorphism** $\iff$ $\exists f$ such that $h = \ker f$.*
- *h is a **normal epimorphism** $\iff$ $\exists f$ such that $h = \operatorname{coker} f$.*

So a monomorphism is regular, iff it is the equalizer of a certain pair of parallel arrows. An epimorphism is regular, iff it is a coequalizer. Normal monomorphisms are kernels, whence they are special regular monomorphisms. Dually, normal epimorphisms being cokernels are special epimorphims.

4.13 Example.

1. *Let A stand for one of $\mathcal{SET}$, $\mathcal{TOP}$, $\mathcal{GROUP}$, ${}_\Lambda\mathcal{MOD}$, and consider morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Then the set $\{x \in X \mid fx = gx\}$ equipped with the appropriate structure - together with inclusion - gives $\mathrm{Eq}(f,g)$. This shows that kernels in $\mathcal{GROUP}$ and ${}_\Lambda\mathcal{MOD}$ have their usual meaning.*
2. *Let A be one of $\mathcal{SET}$, $\mathcal{TOP}$, and f, g as before. Let $\rho \subseteq Y \times Y$ denote the smallest equivalence relation that containes $\{(fx, gx) \mid x \in X\}$, and let $\pi\colon Y \longrightarrow Y/\rho$. Then $(\pi, Y/\rho) = \mathrm{coEq}(f,g)$.*
3. *The same construction applies for $\mathcal{GROUP}$ and ${}_\Lambda\mathcal{MOD}$, when ρ denotes the smallest congruence relation containing $\{(fx, gx) \mid x \in X\}$. If we set $g = 0$ in ${}_\Lambda\mathcal{MOD}$, then, since $[0]_\rho = \mathrm{im} f$, we learn that $\mathrm{coker} f = Y/\mathrm{im} f$, as is usual for modules.*
4. *Consider a monomorphism $S \xrightarrow{m} G$ of groups and let $H = \mathrm{im}\, m$. Let $X := G/H \cup \{\star\}$ be the set of left cosets enriched by a new object $\star$. Let K denote the symmetric group on X and write $\sigma := (H, \star)$ for the transposition toggling H and $\star$. We define maps $\varphi, \psi\colon G \longrightarrow K$ by*

   $$\varphi(g)(g'H) = gg'H, \quad \varphi(g)(\star) = \star, \quad \psi(g) = \sigma\varphi(g)\sigma.$$

   These are homomorphisms of groups. Obviously $H = \{g \mid \varphi(g) = \psi(g)\}$, that is, $m = \mathrm{Eq}(\varphi, \psi)$. Consequently every monomorphism in $\mathcal{GROUP}$ is regular.

4.14 Proposition.

- *Every regular monomorphism is extremal.*
- *Every regular epimorphism is extremal.*

Proof. Assume $a \xrightarrow{e} b$ is a regular epimorphism. $\exists r, s$ such that $e = \mathrm{coEq}(r,s)$. Let $e = m\varphi$ with a monomorphism m. Then $\varphi r = \varphi s$, hence there is a unique h such that $he = \varphi$. Thus $mhe = m\varphi = e = 1e$, so $mh = 1$ follows. Consequently m is a retraction. By Proposition 1.24, m is an isomorphism. □

4.15 Proposition.

For any square the following are equivalent:

$$\begin{array}{ccc} c & \xrightarrow{p} & a \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ a & \xrightarrow{f} & b \end{array}$$

1. *The square is both, a pullback and a pushout.*
2. *(p, q) is a congruence relation of f and f is a coequalizer of p, q.*
3. *(p, q) is a congruence relation of f and f is regular epi.*

A monomorphism equivalent to a regular monomorphism is regular. Thus we have the notion of a **regular subobject**. Similarly we obtain **normal subobjects** which are also called **kernels** or **kernel objects**. Dually we have the notion of a **regular quotient object** and a **normal quotient object (Cokernel object)**.

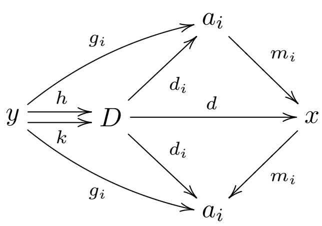
4.16 Example. *From Example 4.13 we see that in $\mathcal{GROUP}$ every subobject of an object is regular. Of course there are subobjects that are not normal, as a group may contain subgroups which are not invariant under inner automorphisms.*

4.17 Proposition. *let (a_i, m_i) be a family of subobjects of an object x. Then*

$$\lim_{\leftarrow}\left(\begin{array}{ccc} a_i & \searrow^{m_i} & \\ \vdots & & x \\ a_i & \nearrow_{m_i} & \end{array}\right) = \bigcap_i (a_i, m_i).$$

Precisely, if the pullback exists, then the intersection exists and both are equal.

Proof. Let $(D, (D \xrightarrow{d_i} a_i)_i)$ be the multiple pullback, let $d = m_i d_i$. Consider morphisms $y \underset{k}{\overset{h}{\rightrightarrows}} D$ with $dh = dk$. Furthermore set $g_i := d_i h$.



We have $m_i d_i = m_j d_j$ and so $m_i g_i = m_j g_j$ for all indices i, j. There is a unique $\varphi\colon y \longrightarrow D$ such that $d_i \varphi = g_i$ all i. From $dk = dh$ we get $m_i d_i k = m_i d_i h$, whence $d_i k = d_i h = g_i$. Consequently $h = k$. This shows that (D, d) is a subobject of x. As it is even the limit of the diagram, it must be the intersection. $\square$