Exercises discussed on June 28, 2011

(HW50) Prove for the cycle indicator polynomial $C(G, X)$ that

$$C(G,X) = \frac{1}{|\overline{G}|} \sum_{g \in \overline{G}} z_1^{a_1(g)} \cdots z_n^{a_n(g)}.$$

(HW51) Show that

$$C(C_n, [n]) = \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) z_{n/d}^d.$$

(HW52) Verify that

$$C\left(S_4, \binom{[n]}{2}\right) = \tfrac{1}{24}\left(z_1^6 + 9z_1^2 z_2^2 + 8z_3^2 + 6z_2 z_4\right).$$

(BP14) Give a direct proof of

$$\left|S_n \backslash\backslash X^N\right| = \binom{n+x-1}{x-1},$$

where N with $|N| = n$ is indistinguishable, X with $|X| = x$ is distinguishable.

(HW53) Prove the weighted form of the Cauchy-Frobenius lemma: Let $M =_G M$, R be a commutative ring containing the rational numbers, and $w : M \to R$ such that $w(gx) = w(x)$ for all $g \in G$ and $x \in M$. Let furthermore T be an orbit transversal, i.e., $M = \dot{\bigcup}_{t \in T} G(t)$, then

$$\sum_{t \in T} w(t) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in M_g} w(x) = \frac{1}{|\overline{G}|} \sum_{g \in \overline{G}} \sum_{x \in M_g} w(x).$$