Exercises discussed on April 5, 2011

(BP4) Show for $k \geq 0$ that

$$\sum_{n=k}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(BP5) Show for $k \geq 0$ that

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\mathrm{e}^x - 1\right)^k.$$

(HW16) Prove $x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}$ by induction.

(BP6) Define the Bernoulli numbers as $B_n = \sum_{k=0}^{n} S(n,k)$ for $n \geq 0$. Show that

$$B_{n+1} = \sum \binom{n}{k} B_k, \qquad n \geq 0.$$

(HW17) Load the Mathematica package Combinatorica and find and test the procedures for

- a) (un)ranking elements of Lists(n)
- b) generating permutations at random
- c) generating labeled trees at random
- d) finding the Prüfer code and its inverse
- e) computing $S(n,k)$, $c(n,k)$ and $s(n,k)$

(HW18) Prove for $n,k \geq 1$ that $s(n,k) = -(n-1)s(n-1,k) + s(n-1,k-1)$.

(HW19) Prove for $n,k \geq 0$ that $c(n,k) = (-1)^{n-k}s(n,k)$.

(HW20) Prove for $n,k \geq 1$ that $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$.

(HW21) Prove for $n,k \geq 1$ that $c(n,k)$ is the number of $\pi \in S_n$ having exactly k cycles.
Hint: Show that both sides satisfy the same recurrence.