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Commutative Algebra & Algebraic Geometry
SS 2010

(8) This exercise is used in the proof of Theorem 2.2.8 about the determination of the radical of a 0-dimensional ideal. It appears originally as Exercise 9 in [CLO98], p.43. Suppose we have an ideal $I \subset K[x_1, \ldots, x_n]$, and let $p = (x_1 - a_1) \cdots (x_1 - a_d)$, where $a_1, \ldots, a_d$ are distinct. The goal of this exercise is to prove that

$$I + \langle p \rangle = \bigcap_{j=1}^{d} \big(I + \langle x_1 - a_j \rangle\big) \ .$$

(a) Prove that $I + \langle p \rangle \subset \bigcap_j (I + \langle x_1 - a_j \rangle)$.
(b) Let $p_j = \prod_{i \neq j} (x_1 - a_i)$. Prove that $p_j \cdot (I + \langle x_1 - a_j \rangle) \subset I + \langle p \rangle$.
(c) Show that $p_1, \ldots, p_d$ are relatively prime, and conclude that there are polynomials $h_1, \ldots, h_d$ such that $1 = \sum_j h_j p_j$.
(d) Prove that $\bigcap_j (I + \langle x_1 - a_j \rangle) \subset I + \langle p \rangle$. Hint: Given h in the intersection, write $h = \sum_j h_j p_j h$ and use part b.

(9) Consider the 0-dimensional ideal $I \subset \mathbb{C}[x, y]$ generated by

$$\begin{aligned} &xy^2 - 2xy - y^2 + x + 2y - 1, \\ &y^3 - y^2 + yx - x - y + 1, \\ &x^2 + y^2 - 1. \end{aligned}$$

Compute the normed reduced Gröbner basis for $\sqrt{I}$ w.r.t. the graduated lexicographic term ordering with $x > y$.

(10) Prove Lemma 3.1.3.