Fundamentals of Numerical Analysis and Symbolic Computation
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Rational Algebraic Curves

Theory and Application

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Contents

What is a rational algebraic curve ?

Some plane algebraic curves can be expressed by means of rational parametrizations, i.e. pairs of univariate rational functions.

For instance, the tacnode curve defined in $\mathbb{A}^2(\mathbb{C})$ by the polynomial equation

$$f(x,y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$$

can be represented, for instance, as

$$\left\{ \left(\frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \quad \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \right) \;\middle|\; t \in \mathbb{C} \right\}.$$

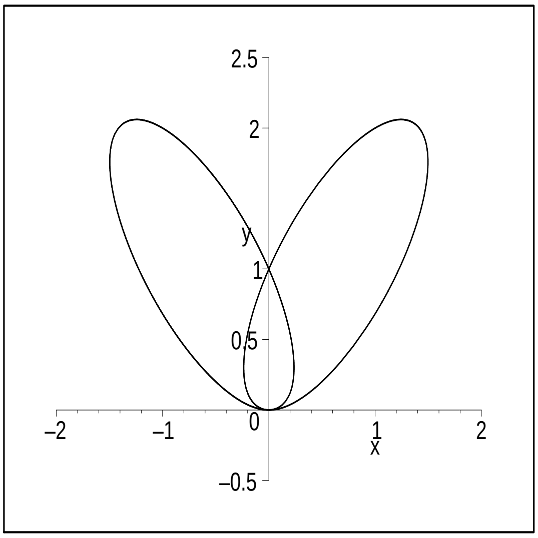

Figure 1: Tacnode curve

However, not all plane algebraic curves can be rationally parametrized, for instance the curve defined by

$$x^3 + y^3 = 1$$

Definition 1.1: *The affine curve* $\mathcal{C}$ *in* $\mathbb{A}^2(K)$ *defined by the square–free polynomial* $f(x, y)$ *is* rational *(or* parametrizable*) if there are rational functions* $\chi_1(t), \chi_2(t) \in K(t)$ *such that*

(1) for almost all $t_0 \in K$ *(i.e. for all but a finite number of exceptions) the point* $(\chi_1(t_0), \chi_2(t_0))$ *is on* $\mathcal{C}$*, and*

(2) for almost every point $(x_0, y_0) \in \mathcal{C}$ *there is a* $t_0 \in K$ *such that* $(x_0, y_0) = (\chi_1(t_0), \chi_2(t_0))$*.*

In this case $(\chi_1(t), \chi_2(t))$ *is called an* affine rational parametrization *of* $\mathcal{C}$*.*
We say that $(\chi_1(t), \chi_2(t))$ *is in* reduced form *if the rational functions* $\chi_1(t), \chi_2(t)$ *are in reduced form; i.e. if for* $i = 1, 2$ *the gcd of the numerator and the denominator of* χ_i *is trivial.*

Definition 1.2: *The projective curve* $\mathcal{C}$ *in* $\mathbb{P}^2(K)$ *defined by the square–free homogeneous polynomial* $F(x, y, z)$ *is* rational *(or* parametrizable*) if there are polynomials* $\chi_1(t), \chi_2(t), \chi_3(t) \in K[t]$, $\gcd(\chi_1, \chi_2, \chi_3) = 1$, *such that*

(1) for almost all $t_0 \in K$ *the point* $(\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$ *is on* $\mathcal{C}$, *and*

(2) for almost every point $(x_0 : y_0 : z_0) \in \mathcal{C}$ *there is a* $t_0 \in K$ *such that* $(x_0 : y_0 : z_0) = (\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$.

In this case, $(\chi_1(t), \chi_2(t), \chi_3(t))$ *is called a* projective rational parametrization *of* $\mathcal{C}$.

Some basic facts

Fact 1: *The notion of rational parametrization can be stated by means of rational maps. More precisely, let* $\mathcal{C}$ *be a rational affine curve and* $\mathcal{P}(t) \in K(t)^2$ *a rational parametrization of* $\mathcal{C}$. *The parametrization* $\mathcal{P}(t)$ *induces the rational map*

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t), \end{array}$$

and $\mathcal{P}(\mathbb{A}^1(K))$ *is a dense (in the Zariski topology) subset of* $\mathcal{C}$. *Sometimes, by abuse of notation, we also call this rational map a rational parametrization of* $\mathcal{C}$.

Fact 2: *Every rational parametrization $\mathcal{P}(t)$ defines a monomorphism from the field of rational functions $K(\mathcal{C})$ to $K(t)$ as follows:*

$$\begin{array}{rccc} \varphi : & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

Fact 3: *Every rational curve is irreducible.*

Fact 4: *Let $\mathcal{C}$ be an irreducible affine curve and $\mathcal{C}^*$ its corresponding projective curve. Then $\mathcal{C}$ is rational if and only if $\mathcal{C}^*$ is rational. Furthermore, a parametrization of $\mathcal{C}$ can be computed from a parametrization of $\mathcal{C}^*$ and vice versa.*

Fact 5: *Let $\mathcal{C}$ be an affine rational curve over K, $f(x, y)$ its the defining polynomial, and*

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$$

a rational parametrization of $\mathcal{C}$. Then, there exists $r \in \mathbb{N}$ such that

$$\mathrm{res}_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y)) = (f(x, y))^r.$$

Fact 6: *An irreducible curve $\mathcal{C}$, defined by $f(x, y)$, is rational if and only if there exist rational functions $\chi_1(t), \chi_2(t) \in K(t)$, not both constant, such that $f(\chi_1(t), \chi_2(t)) = 0$. In this case, $(\chi_1(t), \chi_2(t))$ is a rational parametrization of $\mathcal{C}$.*

Fact 7: *An irreducible affine curve $\mathcal{C}$ is rational if and only if the field of rational functions on $\mathcal{C}$, i.e. $K(\mathcal{C})$, is isomorphic to $K(t)$ (t a transcendental element).*

Fact 8: *An affine algebraic curve $\mathcal{C}$ is rational if and only if it is birationally equivalent to K (i.e. the affine line $\mathbb{A}^1(K)$).*

Fact 9: *If an algebraic curve $\mathcal{C}$ is rational then* $\text{genus}(\mathcal{C}) = 0$.

Proper parametrizations

Definition 2: *An affine parametrization* $\mathcal{P}(t)$ *of a rational curve* $\mathcal{C}$ *is* proper *if the map*

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t) \end{array}$$

is birational, or equivalently, if almost every point on $\mathcal{C}$ *is generated by exactly one value of the parameter* t.
We define the inversion *of a proper parametrization* $\mathcal{P}(t)$ *as the inverse rational mapping of* $\mathcal{P}$, *and we denote it by* $\mathcal{P}^{-1}$.

Lüroth's Theorem: *Let* $\mathbb{L}$ *be a field (not necessarily algebraically closed),* t *a transcendental element over* $\mathbb{L}$*. If* $\mathbb{K}$ *is a subfield of* $\mathbb{L}(t)$ *strictly containing* $\mathbb{L}$*, then* $\mathbb{K}$ *is* $\mathbb{L}$*-isomorphic to* $\mathbb{L}(t)$*.*

Theorem: *Every rational curve can be properly parametrized.*

Theorem: *Let $\mathcal{C}$ be an affine rational curve defined over K with defining polynomial $f(x,y) \in K[x,y]$, and let $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ be a parametrization of $\mathcal{C}$. Then, the following statements are equivalent:*

(1) $\mathcal{P}(t)$ *is proper.*

(2) The monomorphism $\varphi_{\mathcal{P}}$ *induced by* $\mathcal{P}$ *is an isomorphism.*

$$\begin{array}{rccc} \varphi_{\mathcal{P}}: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

(3) $K(\mathcal{P}(t)) = K(t)$.

(4) $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$.

Furthermore, if $\mathcal{P}(t)$ *is proper and* $\chi_1(t)$ *is non-zero, then* $\deg(\chi_1(t)) = \deg_y(f)$*; similarly, if* $\chi_2(t)$ *is non-zero then* $\deg(\chi_2(t)) = \deg_x(f)$.

Example: We consider the rational quintic $\mathcal{C}$ defined by the polynomial $f(x,y) = y^5 + x^2y^3 - 3\,x^2y^2 + 3\,x^2y - x^2$. By the previous theorem, any proper rational parametrization of $\mathcal{C}$ must have a first component of degree 5, and a second component of degree 2. It is easy to check that

$$\mathcal{P}(t) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1}\right)$$

properly parametrizes $\mathcal{C}$. Note that $f(\mathcal{P}(t)) = 0$.

Parametrization algorithm

Theorem (parametrization of conics) *The irreducible projective conic $\mathcal{C}$ defined by the polynomial*

$$F(x, y, z) = f_2(x, y) + f_1(x, y)z$$

(f_i a form of degree i, resp.), has the rational projective parametrization

$$\mathcal{P}(t) = (-f_1(1, t), -tf_1(1, t), f_2(1, t)).$$

Corollary: *Every irreducible conic is rational.*

Example: Let $\mathcal{C}$ be the affine ellipse defined by

$$f(x, y) = x^2 + 2x + 2y^2 = 0 \ .$$

So, a parametrization of $\mathcal{C}$ is

$$\mathcal{P}(t) = (-1 + 2\,t^2, -2\,t, 1 + 2\,t^2).$$

This approach can be immediately generalized to the situation where we have an irreducible projective curve $\mathcal{C}$ of degree d with a $(d-1)$–fold point P. W.l.o.g. we assume that $P = (0:0:1)$. So the defining polynomial of $\mathcal{C}$ is of the form

$$F(x, y, z) = f_d(x, y) + f_{d-1}(x, y)z \ ,$$

where f_i is a form of degree i, respectively. Of course, there can be no other singularity of $\mathcal{C}$, since otherwise the line passing through the two singularities would intersect $\mathcal{C}$ more than d times.

Theorem (curves with point of high multiplicity) *Let $\mathcal{C}$ be an irreducible projective curve of degree d defined by the polynomial $F(x, y, z) = f_d(x, y) + f_{d-1}(x, y)z$ (f_i a form of degree i, resp.), i.e. having a $(d-1)$–fold point at $(0 : 0 : 1)$. Then $\mathcal{C}$ is rational and a rational parametrization is*

$$\mathcal{P}(t) = (-f_{d-1}(1, t), -tf_{d-1}(1, t), f_d(1, t)).$$

Corollary. *Every irreducible curve of degree d with a $(d-1)$-fold point is rational.*

Example: Let $\mathcal{C}$ be the affine quartic curve defined by (see Figure 2)

$$\begin{aligned} f(x,y) &= 1 + x - 15\,x^2 - 29\,y^2 + 30\,y^3 - 25\,xy^2 + x^3y \\ &\quad +35\,xy + x^4 - 6\,y^4 + 6\,x^2y \end{aligned}$$

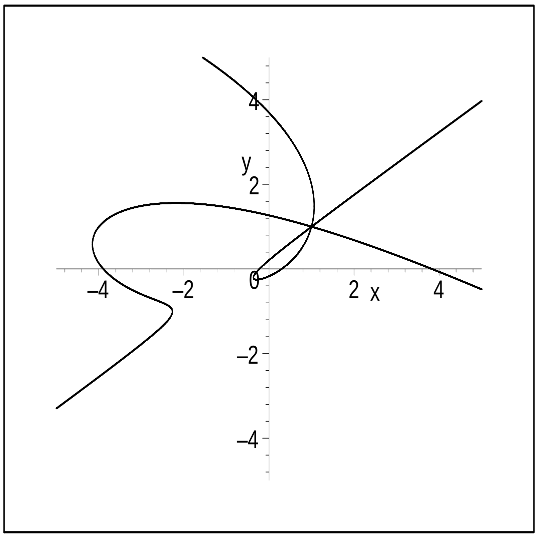

Figure 2: Quartic $\mathcal{C}$

$\mathcal{C}$ has an affine triple point at $(1,1)$. By moving this point to the origin, applying the theorem, and inverting the transformation, we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(\frac{4 + 6\,t^3 - 25\,t^2 + 8\,t + 6\,t^4}{-1 + 6\,t^4 - t}, \frac{4\,t + 12\,t^4 - 25\,t^3 + 9\,t^2 - 1}{-1 + 6\,t^4 - t} \right).$$

Now let $\mathcal{C}$ will be an irreducible projective curve of degree $d > 2$ and genus 0.

Definition: A linear system of curves $\mathcal{H}$ *parametrizes* $\mathcal{C}$ iff

(1) $\dim(\mathcal{H}) = 1$,

(2) the intersection of a generic element in $\mathcal{H}$ and $\mathcal{C}$ contains a non–constant point whose coordinates depend rationally on the free parameter in $\mathcal{H}$,

(3) $\mathcal{C}$ is not a component of any curve in $\mathcal{H}$.

In this case we say that $\mathcal{C}$ *is parametrizable by* $\mathcal{H}$.

Theorem: *Let $F(x, y, z)$ be the defining polynomial of $\mathcal{C}$, and let $H(t, x, y, z)$ be the defining polynomial of a linear system $\mathcal{H}(t)$ parametrizing $\mathcal{C}$. Then, the proper parametrization $\mathcal{P}(t)$ generated by $\mathcal{H}(t)$ is the solution in $\mathbb{P}^2(K(t))$ of the system of algebraic equations*

$$\left.\begin{array}{l} \mathrm{pp}_t(\mathrm{res}_y(F, H)) = 0 \\ \mathrm{pp}_t(\mathrm{res}_x(F, H)) = 0 \end{array}\right\} .$$

Theorem: *Let $\mathcal{C}$ be a projective curve of degree d and genus 0, let $k \in \{d-1, d-2\}$, and let $\mathcal{S}_k \subset \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ be such that $\mathrm{card}(\mathcal{S}_k) = kd - (d-1)(d-2) - 1$. Then*

$$\mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}_k} P)$$

parametrizes $\mathcal{C}$.

Example: Let $\mathcal{C}$ be the quartic over $\mathbb{C}$ (see Figure 3) of equation

$$F(x,y,z) = -2xy^2z-48x^2z^2+4xyz^2-2x^3z+x^3y-6y^4+48y^2z^2+6x^4.$$

The singular locus of $\mathcal{C}$ is

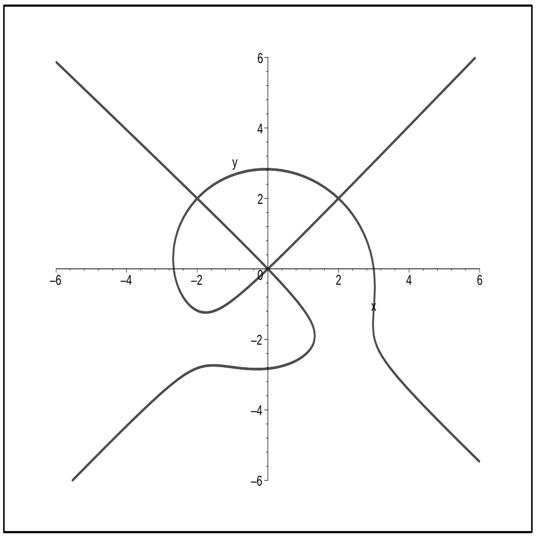

Figure 3: $\mathcal{C}_{\star,z}$

$$\mathrm{Sing}(\mathcal{C}) = \{(0:0:1),(2:2:1),(-2:2:1)\},$$

all three points being double points. Therefore, genus$(\mathcal{C}) = 0$, and hence $\mathcal{C}$ is rational.

We proceed to parametrize the curve. The defining polynomial of $\mathcal{A}_2(\mathcal{C})$ (adjoint curves of degree 2) is

$$H(x,y,z) = (-2\,a_{02} - 2\,a_{20})\,yz + a_{02}y^2 - 2\,a_{11}xz + a_{1,1}xy + a_{20}x^2.$$

We choose a set $\mathcal{S} \subset (\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C}))$ with 1 point, namely $\mathcal{S} = \{(3:0:1)\}$. We compute the defining polynomial of $\mathcal{H} := \mathcal{A}_2(\mathcal{C}) \cap \mathcal{H}(2,Q)$, where $Q = (3:0:1)$. This leads to

$$H(x,y,z) = (-2\,a_{02} - 2\,a_{20})\,yz + a_{02}y^2 - 3\,a_{20}xz + \frac{3}{2}\,a_{20}xy + a_{20}x^2.$$

Setting $a_{02} = 1, a_{20} = t$, we get the defining polynomial

$$H(t,x,y,z) = (-2-2\,t)\,yz + y^2 - 3\,txz + \frac{3}{2}\,txy + tx^2$$

of the parametrizing system. Finally, the solution of the system defined by the resultants provides the following affine parametrization of $\mathcal{C}$

$$\begin{aligned}\mathcal{P}(t) \;=\; \Big(&12\,\tfrac{9\,t^4+t^3-51\,t^2+t+8}{126\,t^4-297\,t^3+72\,t^2+8\,t-36},\\ &-2\,\tfrac{t(162\,t^3-459\,t^2+145\,t+136)}{126\,t^4-297\,t^3+72\,t^2+8\,t-36}\Big)\,.\end{aligned}$$

Further important topics:

- optimal field of parametrization, finding rational points on conics
- making a non-proper parametrization proper (proper reparametrization)
- making a parametrization polynomial
- making a parametrization normal

Application: Solving Diophantine Equations

Curve parametrizations can be used to solve certain types of Diophantine equations. For further details on this application we refer to (Poulakis,Voskos, JSC, 2000 & 2002).
We consider a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ and the curve $\mathcal{C}$ defined by f. If $\mathcal{C}$ cannot be parametrized over $\mathbb{Q}$, then the only integer solutions are the integer singular points of the curve. Otherwise, we compute a rational proper parametrization of $\mathcal{C}$ over $\mathbb{Q}$ in reduced form,

$$\mathcal{P}(t) = \left(\frac{u(t)}{w_1(t)}, \frac{v(t)}{w_2(t)} \right) \in \mathbb{Q}(t).$$

We homogenize the rational functions of the parametrization, say

$$\mathcal{P}^*(t, s) = \left(\frac{U(t, s)}{W_1(t, s)}, \frac{V(t, s)}{W_2(t, s)} \right).$$

Now, because of our assumptions, either $W_1(t,s)$ or $W_2(t,s)$ have at least three different factors. Let us assume w.l.o.g. that W_1 satisfies this property. Then, we compute the resultant $R_1 = \mathrm{res}_t(U(t,1), W_1(t,1))$, and the greatest common divisor, δ_1, of the cofactors of the first column of the Sylvester matrix of $U(t,1), W_1(t,1)$. A similar strategy is applied to $U(1,s), W_1(1,s)$ to get R_2 and δ_2. Next we determine the integer solutions (t,s) with $\gcd(t,s)=1$ and $t \geq 0$, of the Thue equations

$$W_1(t,s) = k,$$

where $k \in \mathbb{Z}$ divides $\mathrm{lcm}(R_1/\delta_1, R_2/\delta_2)$. Let us say that $\mathcal{S}$ is the set of integer solutions of these Thue equations.

Then, the integer singular points of $\mathcal{C}$ and the points in $\{\mathcal{P}^*(t,s) \,|\, (t,s) \in \mathcal{S}\} \cap \mathbb{Z}^2$ are all the integer solutions of the equation $f(x,y) = 0$.

Specific example:
Let n be a positive integer, and let $\mathcal{C}_n$ be the curve defined by

$$f_n(x,y) = x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 - 2ny(x+y).$$

All these curves $\mathcal{C}_n$ are irreducible cubics with a double point at the origin and can be parametrized:
Parametrization of $\mathcal{C}_n$:

$$\mathcal{P}_n(t) = \left(\tfrac{2nt^2+2nt}{t^3-(n-1)t^2-(n+2)t-1}, \tfrac{2nt+2n}{t^3-(n-1)t^2-(n+2)t-1}\right).$$

Now, we consider

$$U(n,t,s) = 2nt^2s + 2nts^2,\ V(n,t,s) = 2nts^2 + 2ns^3,$$
$$W(n,t,s) = t^3 - (n-1)t^2s - (n+2)ts^2 - s^3.$$

Note that in this example, $W_1 = W_2 = W(n,t,s)$. Therefore,

$$\mathcal{P}_n^*(t,s) = \left(\tfrac{2nt^2s+2nts^2}{t^3-(n-1)t^2s-(n+2)ts^2-s^3}, \tfrac{2nts^2+2ns^3}{t^3-(n-1)t^2s-(n+2)ts^2-s^3}\right).$$

We get:

$$\begin{array}{l} R_1 = 8n^3, \quad \delta_1 = 4n^2, \quad R_1/\delta_1 = -2n \\ R_2/\delta_2 = -2n \\ \mathrm{lcm}(R_1/\delta_1, R_2/\delta_2) = 2n \end{array}$$

$$\begin{array}{rl} \mathcal{S} = \{ & (1,0),\, (0,1),\, (1,-1),\, (1,1),\, (1,-2),\, (2,-1), \\ & (1,-n-1),\, (n,1),\, (n+1,-n)\}. \end{array}$$

So

$$\begin{array}{l} \{\mathcal{P}_n^*(t,s) \,|\, (t,s) \in \mathcal{S}\} \cap \mathbb{Z}^2 \ = \\ \qquad \{(0,0) = \mathcal{P}_n^*(1,0),\ \ (0,-2n) = \mathcal{P}_n^*(0,1)\}. \end{array}$$

Since the only singularity of $\mathcal{C}_n$ is $(0,0)$, we deduce that the integer solutions to $f_n(x,y) = 0$ are $(0,0)$ and $(0,-2n)$.

Application: General Solutions of First Order ODEs

This is described in (Feng,Gao, Proc. ISSAC 2004).

Let $F(y, y')$ be a first order irreducible differential polynomial with coefficients in $\mathbb{Q}$. If

$$\overline{y} = \frac{\overline{a}_n x^n + \cdots + \overline{a}_0}{x^m + \overline{b}_{m-1} x^{m-1} + \cdots + \overline{b}_0},$$

is a nontrivial solution of $F(y, y') = 0$, where $\overline{a}_i, \overline{b}_j \in \mathbb{Q}$, and $\overline{a}_n \neq 0$, then (for an arbitrary constant c)

$$\hat{y} = \frac{\overline{a}_n (x+c)^n + \cdots + \overline{a}_0}{(x+c)^m + \overline{b}_{m-1} (x+c)^{m-1} + \cdots + \overline{b}_0},$$

is a general solution of $F(y, y') = 0$. Therefore, the problem of finding a rational general solution is reduced to the problem of finding a nontrivial rational solution.

The polynomial $F(y, y_1) \in \mathbb{Q}[y, y_1]$ defines an algebraic plane curve $\mathcal{C}$. Now, if $\overline{y} = r(x) \in \mathbb{Q}(t)$ is a nontrivial rational solution of $F(y, y') = 0$, then

$$\mathcal{P}(x) = (r(x), r'(x)) \in \overline{\mathbb{Q}}(x)^2$$

can be regarded as a rational parametrization of $\mathcal{C}$. In fact, $\mathcal{P}(x)$ is a proper parametrization of $\mathcal{C}$. Feng,Gao show that given a proper rational parametrization
$\mathcal{P}(x) = (r(x), s(x)) \in \overline{\mathbb{Q}}(x)^2$ of $\mathcal{C}$, the differential equation $F(y, y')$ has a rational solution if and only if one of the following relations

$$ar'(x) = s(x) \quad \text{or} \quad a(x - b)^2 r'(x) = s(x), \tag{$*$}$$

is satisfied, where $a, b \in \overline{\mathbb{Q}}$, and $a \neq 0$. Moreover, if one of the above relations holds, replacing

$$x \text{ by } a(x + c) \qquad \text{or by } (ab(x + c) - 1)/(a(x + c)),$$

respectively, in $y(x) = r(x)$, one obtains a rational general solution of $F(y, y') = 0$, where c is an arbitrary constant.

Specific example:

We consider the differential equation

$$\begin{aligned} F(y,y') &= 229 - 144y + 16y(y')^2 + 16y^4 - 128y^2 + 4y(y')^3 \\ &\quad +4y^3 - 4y^3(y')^2 \\ &\quad -y^2(y')^2 + 6(y')^2 + (y')^3 + (y')^4 = 0. \end{aligned}$$

The curve $\mathcal{C}$ associated to the differential equation is defined by

$$\begin{aligned} F(y,y_1) &= 229 - 144y + 16yy_1^2 + 16y^4 - 128y^2 + 4yy_1^3 + 4y^3 \\ &\quad -4y^3y_1^2 - y^2y_1^2 + 6y_1^2 + y_1^3 + y_1^4. \end{aligned}$$

$\mathcal{C}$ is rational and a parametrization is

$$(r(x), s(x)) = \left(\frac{x^3 + x^4 + 1}{x^2}, \frac{x^3 + 2x^4 - 2}{x} \right) .$$

Now, we see that

$$\frac{s}{r'} = x^2.$$

Therefore, the second condition in $(*)$ is satisfied with $a = 1, b = 0$. Substituting

$$\frac{ab(x+c) - 1}{a(x+c)} = \frac{-1}{x+c},$$

in $r(x)$ we get the following rational general solution of the differential equation:

$$\hat{y} = \frac{-x - c + 1 + x^4 + 4x^3c + 6x^2c^2 + 4xc^3 + c^4}{(x+c)^2}.$$

Applications in CAGD

Computer aided geometric design (CAGD) is a natural environment for practical applications of algebraic curves and surfaces, and in particular of rational curves and rational surfaces. The widely used Bézier curves and surfaces are typical examples of rational curves and surfaces. Offsetting and blending of such geometrical objects lead to interesting problems.

The notion of an offset is directly related to the concept of an envelope. More precisely, the offset curve, at distance d, to an irreducible plane curve $\mathcal{C}$ is "essentially" the envelope of the system of circles centered at the points of $\mathcal{C}$ with fixed radius d (see Figure 4). Offsets arise in practical applications such as

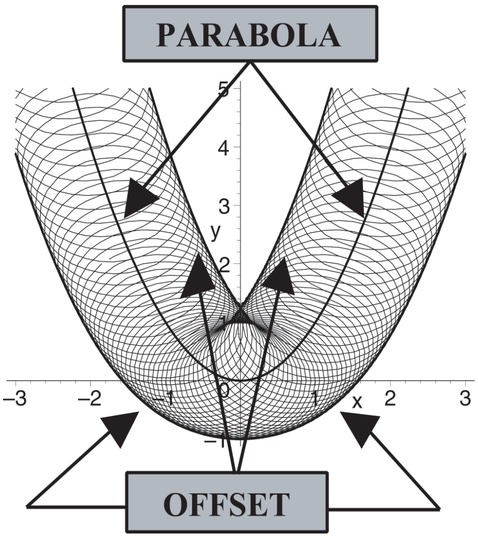

Figure 4: Generation of the offsets to the parabola

tolerance analysis, geometric control, robot path-planning and numerical-control machining problems.

In general the rationality of the original curve is not preserved in the transition to the offset. For instance, while the parabola, the ellipse, and the hyperbola are rational curves (compare Figure 5), the offset of a parabola is rational but the offset of an ellipse or a hyperbola is not rational.

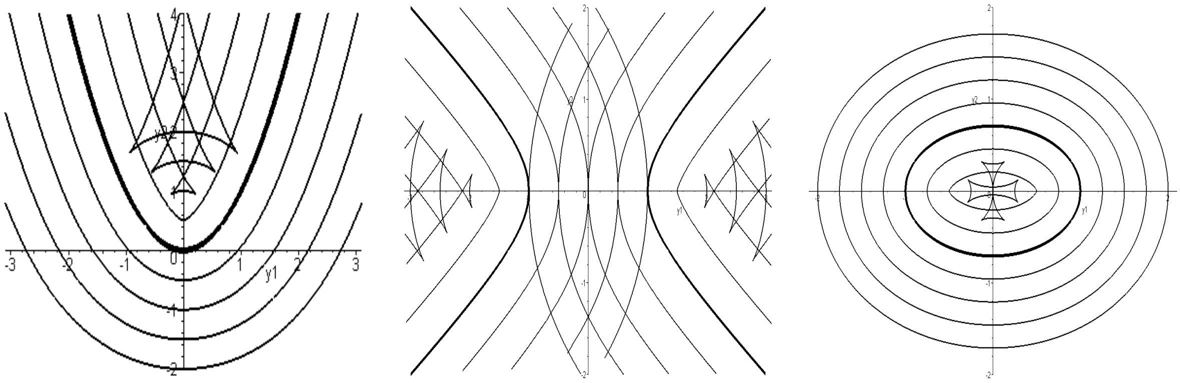

Figure 5: Offsets to the parabola (left), to the hyperbola (center), to the ellipse (right)

Let $\mathcal{C}$ be the original rational curve and let

$$\mathcal{P}(t) = (P_1(t), P_2(t))$$

be a proper rational parametrization of $\mathcal{C}$.
We determine the normal vector associated to the parametrization $\mathcal{P}(t)$, namely

$$\mathcal{N}(t) := (-P_2'(t), P_1'(t)).$$

Note that the offset at distance d basically consist of the points of the form

$$\mathcal{P}(t) \pm \frac{d}{\sqrt{P_1'(t)^2 + P_2'(t)^2}} \mathcal{N}(t).$$

Now we check whether this parametrization satisfies the "rational Pythagorean hodograph condition", i.e. whether

$$P_1^{'}(t)^2 + P_2^{'}(t)^2,$$

written in reduced form, is the square of a rational function in t. If the condition holds, then the offset to $\mathcal{C}$ has two components, and both components are rational. In fact, these two components are parametrized as

$$\mathcal{P}(t) + \frac{d}{m(t)}\mathcal{N}(t), \text{ and } \mathcal{P}(t) - \frac{d}{m(t)}\mathcal{N}(t),$$

where $P_1^{'}(t)^2 + P_2^{'}(t)^2 = a(t)^2/b(t)^2$ and $m(t) = a(t)/b(t)$.
If the rational Pythagorean hodograph condition does not hold, then the offset is irreducible and we may determine its rationality.

Specific example:
We consider as initial curve the parabola of equation $y = x^2$, and its proper parametrization

$$\mathcal{P}(t) = (t, t^2).$$

The normal vector associated to $\mathcal{P}(t)$ is $\mathcal{N}(t) = (-2t, 1)$. Now, we check the rational Pythagorean hodograph condition

$$P_1^{'}(t)^2 + P_2^{'}(t)^2 = 4t^2 + 1,$$

and we observe that $4t^2 + 1$ is not the square of a rational function. Therefore, the offset to the parabola is irreducible. In fact, the offset to the parabola, at a generic distance d, can be parametrized as

$$\left(\frac{(t^2+1-4dt)(t^2-1)}{4t\,(t^2+1)}, \frac{t^6-t^4-t^2+1+32dt^3}{16t^2\,(t^2+1)}\right).$$

The implicit equation of the offset to the parabola is
$-y^2+32x^2d^2y^2-8x^2yd^2+d^2+20x^2d^2-32x^2y^2+8d^2y^2+2yx^2-8yd^2+48x^4d^2-16x^4y^2-48x^2d^4+40x^4y+32x^2y^3-16d^4y^2-32d^4y+32d^2y^3-x^4+8d^4+8y^3-16x^6+16d^6-16y^4=0.$

Reference

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