Chapter 3

Algebraic sets and varieties

3.1 Affine Space and Algebraic Sets

Throughout this chapter let K be a field.

Def. 3.1.1. The *n–dimensional affine space* over K is defined as

$$\mathbb{A}^n(K) := \{\ (a_1, \ldots, a_n) \mid a_i \in K\ \}.$$

If K is clear from context, we simply write $\mathbb{A}^n$. The elements of $\mathbb{A}^n$ are called *points*. $\mathbb{A}^1$ is called the *affine line*, and $\mathbb{A}^2$ is called the *affine plane*. □

Def. 3.1.2. Let $f \in K[x_1, \ldots, x_n]$. A point $P = (a_1, \ldots, a_n) \in \mathbb{A}^n(K)$ is a *root* or *zero* of f iff $f(P) = f(a_1, \ldots, a_n) = 0$.

A subset $V \subseteq \mathbb{A}^n(K)$ is an *affine algebraic set* iff there is a set of polynomials $S \subseteq K[x_1, \ldots, x_n]$ such that

$$V = V(S) = \{\ P \in \mathbb{A}^n(K) \mid f(P) = 0 \text{ for all } f \in S\ \}.$$ □

We list a few facts about affine algebraic sets:

(1) If $S \subseteq K[x_1, \ldots, x_n]$ and $I = \text{ideal}(S) = \langle S \rangle$, then $V(S) = V(I)$. So every affine algebraic set is $V(I)$ for some ideal I in $K[x_1, \ldots, x_n]$. Since the polynomial ring is Noetherian (see Hilbert's Basis Theorem, below), every ideal has a finite basis. So for every affine algebraic set V there is a finite set of polynomials $S = \{f_1, \ldots, f_m\}$ such that $V = V(S) = V(f_1, \ldots, f_m)$. The corresponding system of algebraic equations

$$f_1 = 0, \ldots, f_m = 0$$

is called a *system of defining equations* for V.

(2) If I, J are ideals with $I \subseteq J$, then $V(I) \supseteq V(J)$.

(3) If $\{I_\alpha\}_{\alpha \in A}$ is an arbitrary family of ideals, then

$$V(\bigcup_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha).$$

Thus, the intersection of an arbitrary family of algebraic sets is an algebraic set.

(4) $V(f \cdot g) = V(f) \cup V(g)$ for polynomials f, g. This relation can be generalized to ideals I, J.

$$V(I) \cup V(J) = V(\{f \cdot g | f \in I, g \in J\}).$$

So if $B_I = \{f_1, \ldots, f_m\}$ and $B_J = \{g_1, \ldots, g_p\}$ are finite bases for the ideals I and J, respectively, then $B = \{f_i \cdot g_j | 1 \leq i \leq m, 1 \leq j \leq p\}$ is a finite basis for $I \cdot J$, the product of the ideals I, J, and

$$V(I) \cup V(J) = V(I \cdot J) = V(B).$$

By the way, we also have $V(I) \cup V(J) = V(I \cap J)$.
So every finite union of algebraic sets is an algebraic set.

(5) $V(0) = \mathbb{A}^n(K)$, and $V(1) = \emptyset$.
$V(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\}$ for all $a_i \in K$. So every finite set of points is an algebraic set.

(6) Collecting (3), (4), and (5) we get that $\mathbb{A}^n(K)$ is a topological space if we take the algebraic sets as the closed sets. This topology is called the *Zariski topology.*

Def. 3.1.3. The *Zariski topology* on $\mathbb{A}^n(K)$ is the topology in which the closed sets are exactly the algebraic sets in $\mathbb{A}^n(K)$. □

Some examples of affine algebraic sets:

(1) Linear algebraic sets: they are the solutions of systems of linear equations and are treated in linear algebra.

(2) Hypersurfaces: these are algebraic sets defined by a single equation $f(x_1, \ldots, x_n) = 0$, where f is non-constant.
If f is linear, we have a hyperplane (a plane in $\mathbb{A}^3$, a line in $\mathbb{A}^2$).
Hypersurfaces in $\mathbb{A}^3$ are just called surfaces.
By definition, every algebraic set is the intersection of finitely many hypersurfaces.
Over the field $\mathbb{R}$ a hypersurface can be empty or consist of only finitely many points:

$$\begin{array}{lll} x^2 + y^2 + 1 = 0 & \longrightarrow & \text{no point in } \mathbb{A}^2(\mathbb{R}) \\ x^2 + y^2 = 0 & \longrightarrow & \text{only one point } (0,0)\text{in } \mathbb{A}^2(\mathbb{R}) \end{array}$$

This cannot happen over an algebraically closed field such as $\mathbb{C}$.

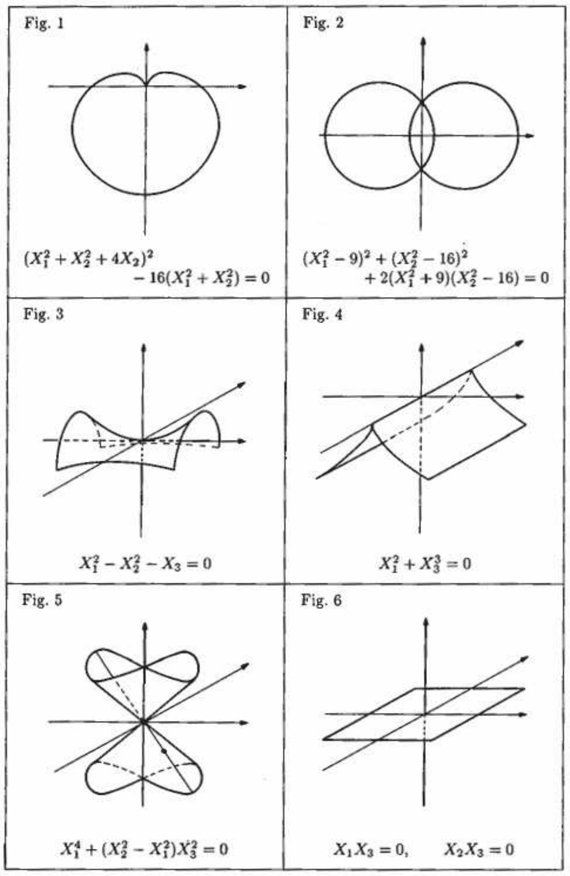

Figure 3.1: from [Kun85]

(3) Plane algebraic curves: a plane algebraic curve $\mathcal{C}$ is a hypersurfaces in $\mathbb{A}^2$, i.e. the set of solutions of $f(x,y)=0$.

(4) Cones: if the defining system of equations consists only of homogeneous polynomials, then the corresponding algebraic set V has the property that for $P \in V$, $P \neq (0,0)$, the whole line connecting P and the origin $O=(0,0)$ is contained in V. Such an algebraic set is called a cone with vertex at the origin.

(5) Product of affine algebraic sets:

$$\begin{array}{ll} V \subseteq \mathbb{A}^n(K) & \text{defined by } f_i(x_1,\dots,x_n)=0,\ i=1,\dots,r \\ W \subseteq \mathbb{A}^m(K) & \text{defined by } g_j(x_1,\dots,x_m)=0,\ i=1,\dots,s \end{array}$$

The product $V \times W \subseteq \mathbb{A}^{n+m}(K)$ is defined by

$$\begin{aligned} f_i(x_1, \ldots, x_n) &= 0, \qquad 1 \le i \le r \\ g_j(y_1, \ldots, y_m) &= 0, \qquad 1 \le j \le s \end{aligned}$$

in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

(6) Parametrizations are points in spaces over rational function fields: let $\mathcal{C} \subset \mathbb{A}^2(K)$ be a curve defined as the set of solutions of $f(x, y) = 0$ over the field K. Let $\mathcal{C}$ be parametrized by $P(t) = (x(t), y(t))$ (compare Example 1.3.). So $f(x(t), y(t)) = 0$, which means that $P(t)$ is a point on the curve $\tilde{\mathcal{C}}$ defined by $f(x, y)$ over the bigger field $\mathbb{A}^2(K(t))$. In fact,

$$\tilde{\mathcal{C}} = \{P(t) \in \mathbb{A}^2(K(t)) \mid f(P(t)) = 0 \} \ .$$

$\tilde{\mathcal{C}}$ contains all the points of $\mathcal{C}$ and also all the parametrizations of $\mathcal{C}$ (or of components thereof; compare Chap. 8).

Theorem 3.1.1. *Let the field K be infinite.*

(a) Let $n \ge 1$. Then for every hypersurface V in $\mathbb{A}^n(K)$ there are infinitely many points in $\mathbb{A}^n(K) \setminus V$, i.e. outside of V.

(b) Let K be algebraically closed and $n \ge 2$. Then every hypersurface in $\mathbb{A}^n(K)$ contains infinitely many points.

Proof: (a) We proceed by induction on n. For $n = 1$ the statement obviously holds. Now consider $n > 1$. The hypersurface V is defined by the non-constant polynomial $f(x_1, \ldots, x_n)$. W.l.o.g. we may assume that x_n actually occurs in f, i.e.

$$f = \sum_{i=0}^{m} g_i(x_1, \ldots, x_{n-1}) x_n^i, \tag{$*$}$$

with $m > 0$ and $g_m \neq 0$.

By the induction hypothesis there is a point $(a_1, \ldots, a_{n-1}) \in \mathbb{A}^{n-1}$ such that $g_m(a_1, \ldots, a_{n-1}) \neq 0$. So $f(a_1, \ldots, a_{n-1}, x_n)$ is a non-vanishing polynomial in $K[x_n]$, having only finitely many roots. Thus, there are infinitely many $a_n \in K$ such that $f(a_1, \ldots, a_{n-1}, a_n) \neq 0$.
(b) Let V be defined by f as in $(*)$. By (a), there are infinitely many points $P = (a_1, \ldots, a_{n-1}) \in \mathbb{A}^{n-1}$ with $g_m(a_1, \ldots, a_{n-1}) \neq 0$. Since K is algebraically closed, for every such point P there is a value $a_n \in K$ such that $f(a_1, \ldots, a_{n-1}, a_n) = 0$. □

We have seen how we can associate a geometric variety to a polynomial ideal. On the other hand, any set of points in space also determines a polynomial ideal, namely the set of polynomials vanishing on these points.

Def. 3.1.4. Let X be a subset of $\mathbb{A}^n(K)$. The set of all polynomials in $K[x_1, \ldots, x_n]$ vanishing on all the points in X form an ideal. This ideal is the *ideal of* X, $I(X)$.

$$I(X) := \{ \ f \in K[x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in X \ \}. \qquad \Box$$

Theorem 3.1.2. *Let K be algebraically closed and $n \geq 1$. Let $H \subset \mathbb{A}^n(K)$ be a hypersurface defined by the polynomial*

$$f = c \cdot f_1^{\alpha_1} \cdot \ldots \cdot f_s^{\alpha_s},$$

where $c \in K^*$, and the f_i are pairwise relatively prime irreducible polynomials. Then $I(H) = \langle f_1 \cdot \ldots \cdot f_s \rangle$.

Proof: Obviously $f_1 \cdot \ldots \cdot f_s \in I(H)$.

So it suffices to show that every $g \in I(H)$ is divisible by all the factors f_i, $1 \leq i \leq s$. Suppose for some i the factor f_i does not divide g. W.l.o.g. we may assume that x_n actually occurs in f_i, i.e.

$$f_i = \sum_{i=0}^{m} g_i(x_1, \ldots, x_{n-1}) x_n^i,$$

with $m > 0$ and $g_m \neq 0$. By Gauss' Lemma, the polynomials f_i and g are also relatively prime in the Euclidean domain $K(x_1, \ldots, x_{n-1})[x_n]$. So for some $h_1, h_2 \in K[x_1, \ldots, x_n]$ and $d \in K[x_1, \ldots, x_{n-1}]^*$ we can write

$$d(x_1, \ldots, x_{n-1}) = h_1(x_1, \ldots, x_n) \cdot f_i(x_1, \ldots, x_n) + h_2(x_1, \ldots, x_n) \cdot g(x_1, \ldots, x_n).$$

By Theorem 3.1.1(a) there is a point $(a_1, \ldots, a_{n-1}) \in \mathbb{A}^{n-1}$ such that

$$d(a_1, \ldots, a_{n-1}) \cdot g_m(a_1, \ldots, a_{n-1}) \neq 0.$$

Choose a value $a_n \in K$ such that $f_i(a_1, \ldots, a_{n-1}, a_n) = 0$. Then $(a_1, \ldots, a_n) \in H$, and therefore $g(a_1, \ldots, a_n) = 0$. This, however, is a contradiction to $d(a_1, \ldots, a_{n-1}) \neq 0$. $\Box$

We list some relations between ideals and algebraic sets.

Lemma 3.1.3. *Let $X, Y \subseteq \mathbb{A}^n(K)$, $S \subseteq K[x_1, \ldots, x_n]$.*

(a) *If $X \subseteq Y$ then $I(X) \supseteq I(Y)$.*

(b) $I(\emptyset) = K[x_1, \ldots, x_n]$.
If K is infinite, then $I(\mathbb{A}^n) = \langle 0 \rangle$.
$I(\{(a_1, \ldots, a_n)\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ *for all $a_i \in K$.*

(c) $I(V(S)) \supseteq S$.
$V(I(X)) \supseteq X$.

(d) $V(I(V(S))) = V(S)$.
$I(V(I(X))) = I(X)$.

(e) *$I(X)$ is a radical ideal.*

The proof is left to the reader as an exercise.

3.2 Hilbert's Basis Theorem

We have already used the fact that every ideal in the polynomial ring $K[x_1, \ldots, x_n]$ is finitely generated. In this section we give a proof of this fact.

Def. 3.2.1. A commutative ring with identity R is called a *Noetherian ring* iff the *basis condition* holds in R, i.e. every ideal in R is finitely generated. □

Lemma 3.2.1. *A commutative ring with identity R is Noetherian if and only if there are no infinite properly ascending chains of ideals in R. I.e., if*

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq R,$$

then there is an index k such that

$$I_k = I_{k+1} = \ldots \ . \qquad \square$$

Proof: Suppose that R is Noetherian. Let

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

be an ascending chain of ideals in R. Consider

$$I := \bigcup_{i=0}^{\infty} I_i \ .$$

I is an ideal in R, so it has a finite basis. This basis must be contained in some I_k; so

$$I_k = I_{k+1} = \cdots \ .$$

On the other hand, suppose that an ideal I in R does not have a finite basis.
Choose a non-zero element $r_0 \in I$; then $I_0 := \langle r_0 \rangle \neq I$.
Choose $r_1 \in I \setminus I_0$; then $I_1 := \langle r_0, r_1 \rangle \neq I$.
This process can be continued indefinitely, yielding an infinite properly ascending chain of ideals in R. □

Theorem 3.2.2. (Hilbert's Basis Theorem) *If R is a Noetherian ring then also the ring of polynomials $R[x]$ is Noetherian.*

Proof: Let I be an ideal in $R[x]$. We have to show that I has a finite basis.

For $f(x) = a_0 + a_1x + \ldots + a_dx^d \in R[x]^*$, $a_d \neq 0$, we call a_d the *leading coefficient* of f, $\text{lc}(f)$, and a_dx^d the *leading term* of f, $\text{lt}(f)$. The leading coefficient of 0 is 0.

Let J be the set of all leading coefficients of polynomials in I. J is an ideal in R, and therefore has a finite basis. Let $f_1, \ldots, f_k \in I$ be such that their leading coefficients generate J, i.e.

$$J = \langle \text{lc}(f_1), \ldots, \text{lc}(f_k) \rangle.$$

Let N be the highest degree of the f_i's,

$$N = \max_{1 \le i \le k} \deg(f_i).$$

For every $m, 0 \le m < N$, let J_m be the ideal in R consisting of the leading coefficients of all polynomials $f \in I$ with $\deg(f) \le m$. Let $\{f_{mj} | 1 \le j \le k_m\}$ be a finite set of polynomials in I with $\deg(f_{mj}) \le m$, such that J_m is generated by the leading coefficients of the f_{mj}, i.e.

$$J_m = \langle \mathrm{lc}(f_{m1}), \ldots, \mathrm{lc}(f_{mk_m}) \rangle.$$

Now let

$$I' := \langle \{f_1, \ldots, f_k\} \cup \bigcup_{0 \le m < N} \{f_{mj} | 1 \le j \le k_m\} \rangle.$$

We show that $I' = I$, so I has a finite basis.

Obviously $I' \subseteq I$. Suppose that I' is a proper subset of I. Let g be an element of least degree in $I \setminus I'$.

Case $\deg(g) \ge N$: There are polynomials q_i such that

$$\mathrm{lt}(\sum q_i f_i) \;=\; \mathrm{lt}(g).$$

So also $g - \sum q_i f_i \in I \setminus I'$ and $\deg(g - \sum q_i f_i) < \deg(g)$, in contradiction to the minimality of $\deg(g)$.

Case $\deg(g) < N$: Let $m = \deg(g)$. There are polynomials q_j such that

$$\mathrm{lt}(\sum q_j f_{mj}) \;=\; \mathrm{lt}(g).$$

So also $g - \sum q_j f_{mj} \in I \setminus I'$ and $\deg(g - \sum q_j f_{mj}) < \deg(g)$, in contradiction to the minimality of $\deg(g)$.

In any case we see that such a g cannot exist, i.e. $I = I'$ and I is finitely generated. □

Corollary. *For any n, $K[x_1, \ldots, x_n]$ is a Noetherian ring.*

Proof: K has only two ideals, namely $\langle 0 \rangle, \langle 1 \rangle$. Both are obviously finitely generated. The statement follows from the Theorem by induction on n. □

3.3 Irreducible Components of Algebraic Sets

Def. 3.3.1. An algebraic set $V \subseteq \mathbb{A}^n$ is *reducible* iff there are algebraic sets V_1, V_2 different from V such that $V = V_1 \cup V_2$. Otherwise V is *irreducible*. An irreducible algebraic set is also called a *variety*. □

Theorem 3.3.1. *An algebraic set V is irreducible if and only if $I(V)$ is a prime ideal.*

Proof: "$\Longrightarrow$": Suppose $I(V)$ is not prime. Then there are polynomials f_1, f_2 such that $f_1 \cdot f_2 \in I(V)$ but $f_1, f_2 \notin I(V)$. So $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$, and $V \cap V(f_i) \neq V$ for $i = 1, 2$. Thus, V is reducible.
"$\Longleftarrow$": Suppose $V = V_1 \cup V_2$, where $V_i \neq V$ for $i = 1, 2$. By Lemma 3.1.3(d), also $I(V_i) \neq I(V)$ for $i = 1, 2$. Let $f_i \in I(V_i) \setminus I(V)$ for $i = 1, 2$. Then $f_1 \cdot f_2 \in I(V)$, and therefore $I(V)$ is not prime. □

An algorithm for decomposing an algebraic set V into a finite union of irreducible algebraic sets could proceed as follows: first we decompose V into sets V_1, V_2. Next we decompose V_1 and V_2, and so on. We will reach a finite decomposition if this algorithm terminates. This is a consequence of the following theorem.

Theorem 3.3.2. *Let $\mathcal{S}$ be a non-empty set of ideals in the Noetherian ring R. Then $\mathcal{S}$ contains a maximal element, i.e. there is an $I \in \mathcal{S}$ such that for all other ideals $J \in \mathcal{S}$ we have $I \not\subset J$.*

Proof: Choose an ideal $I_0 \in \mathcal{S}$, and set $\mathcal{S}_0 := \mathcal{S}$. Now let

$$\mathcal{S}_1 := \{\ I \in \mathcal{S} \mid I_0 \subset I \text{ and } I_0 \neq I\ \}.$$

If $\mathcal{S}_1 \neq \emptyset$, then choose an ideal $I_1 \in \mathcal{S}_1$ and let

$$\mathcal{S}_2 := \{\ I \in \mathcal{S} \mid I_1 \subset I \text{ and } I_1 \neq I\ \}.$$

This process is continued as long as $\mathcal{S}_m \neq \emptyset$. The proof is complete if we can show that for some m the set $\mathcal{S}_m$ is empty.
Suppose $\mathcal{S}_m \neq \emptyset$ for all m. Let

$$I := \bigcup_{m=0}^{\infty} I_m\ ,$$

an ideal in R. Let $\{f_1, \ldots, f_r\}$ be a finite basis of I. For a sufficiently big m we have $f_i \in I_m$ for all $1 \leq i \leq r$. So $I = I_m$ and therefore $I_{m+1} = I_m$, a contradiction. □

Also the converse is true; see [ZaS58] I, p.199.

Corollary. *Every non-empty family $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of algebraic sets in $\mathbb{A}^n$ contains a minimal element (w.r.t. to set inclusion "$\subset$").*

Proof: Let $I(V_{\alpha_0})$ be a maximal element in $\{I(V_\alpha)\}_{\alpha \in A}$. Then V_{α_0} is minimal in $\mathcal{V}$. □

Theorem 3.3.3. *Let V be an algebraic set in $\mathbb{A}^n$. Then there is a unique decomposition, up to permutation of the components, of V into irreducible algebraic sets $V_1, \ldots, V_m$ such that*

$$V = V_1 \cup \ldots \cup V_m \quad \text{and} \quad V_i \not\subseteq V_j \text{ for } i \neq j.$$

Proof: (a) Existence of decomposition: Let

$$\mathcal{V} := \{ \, V \subseteq \mathbb{A}^n \mid \; V \text{ is algebraic and } V \text{ is not the union of finitely many irreducible algebraic sets } \}.$$

We want to show that $\mathcal{V} = \emptyset$.

If this is not the case, then $\mathcal{V}$ contains a minimal element, say $\overline{V}$. Since $\overline{V} \in \mathcal{V}$, $\overline{V}$ can be decomposed into $\overline{V} = V_1 \cup V_2$, $V_i \neq \overline{V}$ for $i = 1, 2$. Because of the minimality of $\overline{V}$, V_i cannot be in $\mathcal{V}$, so $V_i = V_{i1} \cup \ldots \cup V_{im_i}$ for V_{ij} irreducible. But then

$$V = \bigcup_{i,j} V_{ij},$$

a contradiction.

(b) Uniqueness: In the decomposition $V = V_1 \cup \ldots \cup V_m$ eliminate all components which are properly contained in another component and also double occurrences of components. The resulting decomposition is *reduced*.

Now consider two reduced decompositions

$$V = V_1 \cup \ldots \cup V_m$$

and

$$V = W_1 \cup \ldots \cup W_l.$$

Then

$$V_i = V \cap V_i = \bigcup_{j=1}^{l} (W_j \cap V_i).$$

Because of the irreducibility of the V_i, W_j, every $V_i \subseteq W_{j(i)}$ for some $j(i)$, and on the other hand $W_{j(i)} \subseteq V_k$ for some k. This is only possible for $V_i = W_{j(i)} = V_k$, i.e. $i = k$. Thus, every V_i is equal to some $W_{j(i)}$.

In the same way, we can show that every W_j is equal to some $V_{i(j)}$. □

Def. 3.3.2. Let $V \subseteq \mathbb{A}^n$ be an algebraic set. Let $V = V_1 \cup \ldots \cup V_m$ be the unique decomposition guaranteed by Theorem 3.3.3. This decomposition is called the *decomposition of V into irreducible components.* □

In Section 4.3 we will compare this result on decompositon of algebraic sets with primary decomposition of polynomial ideals. The situation for primary decomposition is much more complicated.

Theorem 3.3.4. *If K is infinite, then $\mathbb{A}^n(K)$ is irreducible.*

Proof: Suppose $\mathbb{A}^n(K)$ were reducible, and $\mathbb{A}^n(K) = V_1 \cup V_2$ a decomposition. Consider non-zero polynomials $f_1 \in I(V_1) \setminus I(V_2)$, $f_2 \in I(V_2) \setminus I(V_1)$. $f_1 \cdot f_2 \in I(V_1) \cap I(V_2)$, so $0 \neq f_1 \cdot f_2$ vanishes on all points of $\mathbb{A}^n(K)$. This is a contradiction to Theorem 3.1.1. $\square$

We will take the affine plane $\mathbb{A}^2(K)$ as an example and give a complete classification of the algebraic subsets of the plane. Because of Theorem 3.3.3 it suffices to classify the irreducible algebraic sets. All others are constructed from these components.

Theorem 3.3.5. *Let $f, g \in K[x, y]$, f and g relatively prime. Then $V(f, g) = V(f) \cap V(g)$ is a finite set of points.*

Proof: f and g are relatively prime in $K[x][y]$, so by Gauss' Lemma they are also relatively prime in $K(x)[y]$. But $K(x)[y]$ is a Euclidean domain, so we can write the gcd as a linear combination

$$1 = r \cdot f + s \cdot g,$$

for some $r, s \in K(x)[y]$. After eliminating the denominators from this equation, we get

$$d = a \cdot f + b \cdot g,$$

for some $d \in K[x], a, b \in K[x, y]$.

Now if $(c_1, c_2) \in V(f, g)$, then $d(c_1) = 0$. But d has only finitely many roots. So there are only finitely many possible values for the x–coordinate of points in $V(f, g)$. By an analogous consideration we determine that there are only finitely many possible values for the y–coordinate of points in $V(f, g)$. $\square$

Corollary. *If $f(x, y)$ is irreducible in $K[x, y]$ and $V(f)$ is infinite, then $I(V(f)) = \langle f \rangle$ and $V(f)$ is irreducible.*

Proof: If $g \in I(V(f))$, then $V(f, g) = V(f)$ is infinite. So, by the Theorem, f must divide g, and therefore $g \in \langle f \rangle$. The irreducibility of $V(f)$ follows from Theorem 3.3.1. $\square$

Theorem 3.3.6. (classification) *Let the field K be infinite. The following is a complete classification of the irreducible algebraic subsets of $\mathbb{A}^2(K)$:*

(a) $\mathbb{A}^2(K)$ *and* $\emptyset$,

(b) *single points,*

(c) *irreducible algebraic curves $V(f)$, where f is an irreducible polynomial and $V(f)$ is infinite.*

Proof: Let V be an irreducible algebraic set in $\mathbb{A}^2(K)$. If V is finite or $I(V) = \langle 0 \rangle$, then V is of type (a) or (b).

Otherwise, $I(V)$ contains a non-constant polynomial f. Since $I(V)$ is prime, is must also contain an irreducible factor of f. So w.l.o.g. let f be irreducible. Now we claim that $I(V) = \langle f \rangle$. To see this, let $h \in I(V) \setminus \langle f \rangle$. h and f are relatively prime, so by Theorem 3.3.5 $V \subset V(f, h)$ is finite. □

Theorem 3.3.7. *Let K be algebraically closed, $f \in K[x, y]$. Let $f = f_1^{\alpha_1} \cdot \ldots \cdot f_s^{\alpha_s}$ be the factorization of f. Then*

(a) $V(f) = V(f_1) \cup \ldots \cup V(f_s)$ *is the decomposition of $V(f)$ into irreducible components, and*

(b) $I(V(f)) = \langle f_1 \cdot \ldots \cdot f_s \rangle$.

Proof: obvious. □

Example 3.3.1. We consider the intersection of a sphere with radius 2 and a cylinder with radius 1 defined by

$$\begin{aligned} f_1 &= x^2 + y^2 + z^2 - 4 = 0 \ , \\ f_2 &= y^2 + z^2 - 1 = 0 \ . \end{aligned}$$

$V = V(f_1, f_2)$ can be decomposed as

$$V = V(x - \sqrt{3}, y^2 + z^2 - 1) \cup V(x + \sqrt{3}, y^2 + z^2 - 1),$$

and these are the irreducible components of V. □