Potential theory on orthogonal polynomials arising from subnormal and hyponormal operators

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Let μ be a finite positive Borel measure having compact and infinite support $\mathcal{S} := \text{supp}(\mu)$ in the complex plane $\mathbb{C}$. Then, the measure μ yields the Lebesgue spaces $L^2(\mu)$ with inner product

$$\langle f, g\rangle_\mu = \int f(z)\overline{g(z)}d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, g\rangle_\mu^{1/2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ denote the sequence of orthonormal polynomials associated with μ. That is, the unique sequence of the form

$$p_n(\mu, z) = \kappa_n(\mu)z^n + \cdots, \quad \kappa_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot)\rangle_\mu = \delta_{m,n}$.

The recovery from moments problem

The (inverse) moment problem

Given the infinite sequence of complex moments

$$\mu_{m,n} := \int z^m \overline{z}^n d\mu(z), \quad m, n = 0, 1, 2, \dots,$$

where μ is a (non-trivial) finite positive Borel measure with compact support on $\mathbb{C}$, find the support $\mathcal{S} := \text{supp}(\mu)$ of μ.

In many applications we are interested in a truncated version of the above:

Given a finite section of the infinite sequence of complex moments $\{\mu_{m.n}\}$ compute an approximation to $\mathcal{S}$.

Existence - A partial result

Theorem (Atzmon, Pacific J. Math., 1975)

Let $\{a_{m,n}\}_{m,n=0}^{\infty}$ be an infinite matrix of complex numbers. Then, $a_{m,n} := \int z^m \overline{z}^n d\mu(z)$, $m, n = 0, 1, 2, \ldots$, holds for some positive Borel measure μ on the closed unit disc, if and only if for any matrix $\{c_{j,k}\}_{j,k=0}^{\infty}$ with only finitely many nonzero entries:

$$\sum_{m,n,j,k=0}^{\infty} a_{m+j,n+k} c_{n,j} \overline{c}_{m,k} \geq 0,$$

and for any sequence $\{w_n\}_{n=0}^{\infty}$ with only finitely many nonzero terms:

$$\sum_{m,n}^{\infty} (a_{m,n} - a_{m+1,n+1}) w_m \overline{w}_n \geq 0,$$

The recovery from moments problem

Uniqueness

The infinite sequence of complex moments

$$\mu_{m,n} := \int z^m \overline{z}^n d\mu(z), \quad m, n = 0, 1, 2, \ldots,$$

defines the measure μ uniqely.

This is a simple consequence of:

- The Riesz representation theorem.
- The complex form of the Stone-Weierstrass theorem.

Question

Are there cases where the analytic moments $\int z^m d\mu(z)$, $m = 0, 1, 2, \ldots$, alone, suffice to define μ uniquely?

The case of Jordan arcs and curves

Theorem (Walsh, 1926)

Assume that Γ *is a bounded Jordan arc and let* $f \in C(\Gamma)$. *Then, for every* $\varepsilon > 0$, *there exists a* $p \in \mathbb{P}[z]$, *such that*

$$\|f(z) - p(z)\|_{L^\infty(\Gamma)} \leq \varepsilon.$$

Similarly, by using conformal mapping it is easy to see that

Theorem (Gaier's book on Approximation, 1987)

Assume that Γ *is a bounded Jordan curve and let* $f \in C(\Gamma)$. *Then, for every* $\varepsilon > 0$, *there exist* p *and* q *in* $\mathbb{P}[z]$, *such that*

$$\|f(z) - \{p(z) + \overline{q(z)}\}\|_{L^\infty(\Gamma)} \leq \varepsilon.$$

Hence, the analytic moments suffice to determine uniquely any positive Borel measure supported on Γ, in both cases.

A counterexample?

Theorem (Sakai, Proc. AMS, 1978)

There exists two distinct Jordan domains G_1 and G_2, such that

$$\int_{G_1} z^m dA(z) = \int_{G_2} z^m dA(z), \quad m = 0, 1, 2, \ldots,$$

where A denotes the area measure.

Note: The area measure is supported on the closure of the domain of definition!

An unicity theorem for measures on outer boundaries

Theorem

Let K be a compact set in the complex plane of positive logarithmic capacity and denote by Ω the component of $\overline{\mathbb{C}} \setminus K$ that contains infinity. Let μ and ν be two positive Borel measures, supported on $\partial\Omega$, such that

$$\int z^m d\mu(z) = \int z^m d\nu(z), \quad m = 0, 1, 2, \ldots,$$

Then $\mu = \nu$.

This is a consequence of Carleson's unicity theorem for measures: (Carleson, Math. Scand.,1964 & Saff and Totik, Logarithmic Potentials, Springer, 1997)

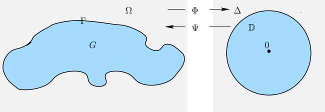
An open problem

Does it hold?

Let K be a compact set in the complex plane of positive logarithmic capacity and denote by Ω the component of $\overline{\mathbb{C}} \setminus K$ that contains infinity, let $f \in C(\partial\Omega)$. Then, for every $\varepsilon > 0$, there exist p and q in $\mathbb{P}[z]$, such that

$$\|f(z) - \{p(z) + \overline{q(z)}\}\|_{L^\infty(\Gamma)} \leq \varepsilon.$$

Recovery of the equilibrium measure: An example



G bounded simply-connected, $\Gamma := \partial G$, $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \boxed{\mathrm{cap}(\Gamma) = b}$$

Theorem (Hille, Analytic Function Theory II, Chelsea, 1962)

Assume that

$$\frac{\Phi'(z)}{\Phi(z)} = \sum_{k=0}^{\infty} \frac{M_k}{z^{k+1}}. \quad \textit{Then,} \quad M_k = \int \zeta^k d\mu_\Gamma,$$

where μ_Γ is the equilibrium measure of Γ.

Recovery of open sets from complex area moments

Theorem (Davis & Pollak, Trans. AMS, 1956)

Let T be a bounded open set which posses exterior points in any neighborhood of any boundary point. Then, the infinite complex moments matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, with respect to the area measure, defines uniquely T.

This leads to applications in 2D geometric tomography, through the Radon transform.

The Arnoldi algorithm for OP's

Let μ be a (non-trivial) finite positive Borel measure with compact support supp(μ) on $\mathbb{C}$ and consider the associated series of orthonormal polynomials

$$p_n(\mu, z) := \kappa_n(\mu) z^n + \cdots, \quad \kappa_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

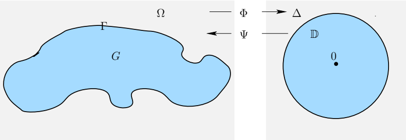
$$\langle f, g \rangle_\mu = \int f(z)\overline{g(z)}d\mu(z), \quad \|f\|_{L^2(\mu)} := \langle f, g \rangle_\mu^{1/2}.$$

Arnoldi Gram-Schmidt (GS) for Orthonormal Polynomials

At the n-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Used by Gragg & Reichel, in Linear Algebra Appl. (1987), for the construction of Szegö polynomials.

Bergman polynomials



$\boxed{\Gamma := \partial G}$ $\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)} dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)} dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots.$$

Ratio asymptotics for $p_n(z)$

Theorem (St, Constr. Approx. 2013)

Assume that Γ *is piecewise analytic without cusps. Then, for any* $z \in \Omega$, *and sufficiently large* $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \le \frac{c_1(\Gamma)}{\sqrt{\mathrm{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Ratio asymptotics for $p_n(z)$

On compact subsets of Ω we have

Theorem (Beckermann & St, Constr. Approx. 2018)

Assume that Γ is piecewise analytic without cusps. Then,

$$\boxed{\sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1+O(1/n)\}},$$

locally uniformly in Ω.

Discovery of a single island

Recovery Algorithm: St, Constr. Approx. 2013

(I) Use the Arnoldi GS to compute $p_0, p_1, \ldots, p_n$.

(II) Compute the coefficients of the Laurent series of the ratio

$$\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \gamma^{(n)} z + \gamma_0^{(n)} + \frac{\gamma_1^{(n)}}{z} + \frac{\gamma_2^{(n)}}{z^2} + \frac{\gamma_3^{(n)}}{z^3} + \cdots . \quad (1)$$

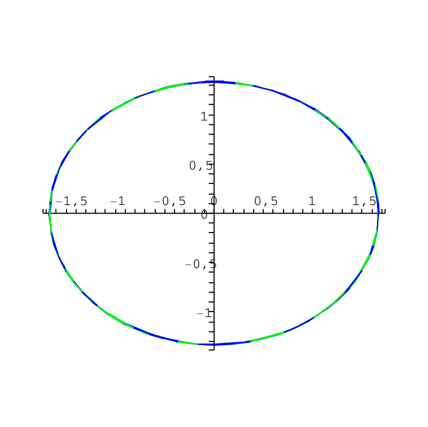
(III) Revert (1) and truncate to obtain

$$\Psi_n(w) := b^{(n)} w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \frac{b_2^{(n)}}{w^2} + \frac{b_3^{(n)}}{w^3} + \cdots + \frac{b_n^{(n)}}{w^n}.$$

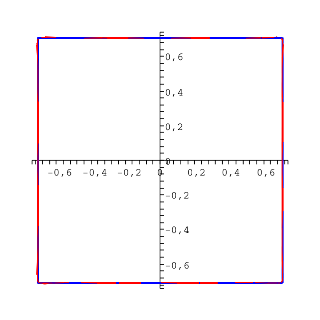
(IV) Approximate Γ by $\widetilde{\Gamma} := \{z : \; z = \Psi_n(e^{it}),\; t \in [0, 2\pi]\}$.

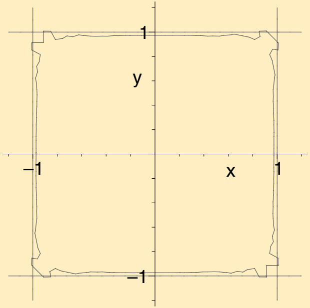
Numerical Examples

Recovery of the canonical ellipse, with $n = 3$.



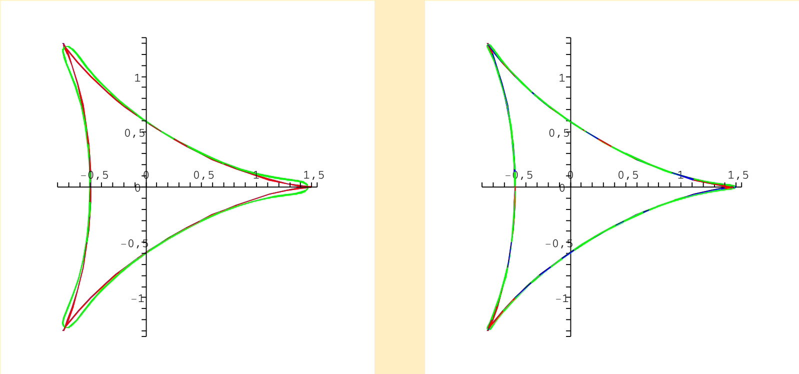
Recovery of the square, with $n = 16$.



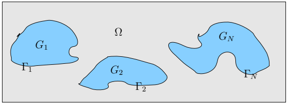


Comparison: The exponential transform algorithm of Gustafsson, He, Milanfar & Putinar, Inverse Problems (2000).

Recovery of the 3-cusped hypocycloid, with $n = 10$ and $n = 20$.



Discovery of an archipelago



Archipelago Recovery Algorithm
(Gustafsson, Putinar, Saff & St, Adv. Math., 2009.)

(I) Use the Arnoldi GS to compute $p_0, p_1, \ldots, p_n$, from $[\mu_{m,k}]_{m,k=0}^{n}$.

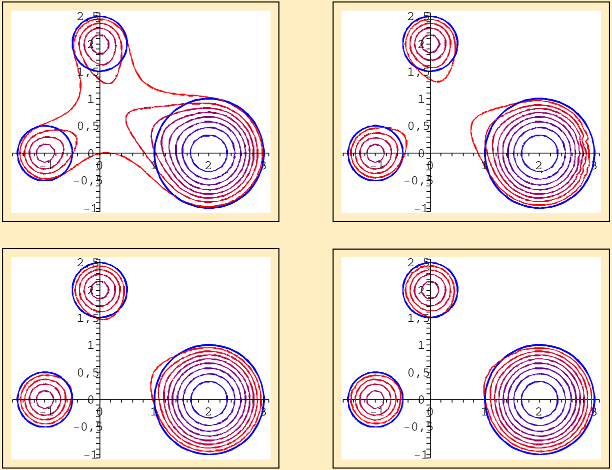
(II) Form the recovery functional

$$\Lambda_n(z) := [K_n(z,z)]^{-1/2} = \left[\sum_{k=0}^{n} |p_k(z)|^2\right]^{-1/2}.$$

(III) Plot the zeros of p_j, for some $1 \leq j \leq n$. (Fejer's Theorem!)

(IV) Plot the level curves of the function $\Lambda_n(x+iy)$, on a suitable rectangular frame for (x,y) that surrounds the plotted zero set.

Recovery of three disks



Level lines of $\Lambda_n(x + iy)$ on $\{(x, y) : -3 \leq x \leq 4, -2 \leq y \leq 3\}$, for $n = 25, 50, 75, 100$.

Shift Operator

Let $\mathcal{P}^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$ and consider the shift operator on $\mathcal{P}^2(\mu)$. That is,

$$S_z : \mathcal{P}^2(\mu) \to \mathcal{P}^2(\mu) \quad \text{with} \quad S_z f = zf.$$

Properties of S_z

(i) S_z defines a subnormal operator on $\mathcal{P}^2(\mu)$.

(ii) $\sigma(S_z) = ?$

(iii) $S_z^*(f) = P(\overline{z}f)$, where P denotes the orthogonal projection from $L^2(\mu)$ to $\mathcal{P}^2(\mu)$.

Proof of (iii): For any $f, g \in \mathcal{P}^2(\mu)$ it holds that

$$\langle S_z^* f, g\rangle = \langle f, S_z g\rangle = \langle f, zg\rangle = \langle \overline{z} f, g\rangle = \langle P(\overline{z} f), g\rangle.$$

Matrix representation for S_z

The shift operator S_z has the following upper Hessenberg matrix representation with respect to the orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$:

$$\mathcal{M} = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & b_{05} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \cdots \\ 0 & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & \cdots \\ 0 & 0 & b_{32} & b_{33} & b_{34} & b_{35} & \cdots \\ 0 & 0 & 0 & b_{43} & b_{44} & b_{45} & \cdots \\ 0 & 0 & 0 & 0 & b_{54} & b_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $\boxed{b_{k,n} = \langle zp_n, p_k \rangle}$ are the Fourier coefficients of $S_z p_n = zp_n$.

Note

The eigenvalues of the $n \times n$ principal submatrix $\mathcal{M}_n$ of $\mathcal{M}$ coincide with the zeros of p_n.

Example: $\mu = dA|_{\mathbb{D}}$

This example shows why modern text books on Functional Analysis or Operators Theory do not refer to infinite matrices: Indeed, in this case we have:

$$p_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, \dots .$$

Therefore, in the matrix representation $\mathcal{M}$ of S_z the only non-zero diagonals are the main subdiagonal, and hence for any $n \in \mathbb{N}$, $\mathcal{M}_n$ is a nilpotent matrix. As a result, the Caley-Hamilton theorem implies:

$$\sigma(\mathcal{M}_n) = \{0\}.$$

This is in sharp contrast to:

$$\sigma_{ess}(\mathcal{M}) = \sigma_{ess}(S_z) = \{w : |w| = 1\}$$

and

$$\sigma(\mathcal{M}) = \sigma(S_z) = \{w : |w| \leq 1\}.$$

Shift Operator on $L^2(\mu)$

Let N_z denote the shift operator on $L^2(\mu)$. That is,

$$N_z : L^2(\mu) \to L^2(\mu) \quad \text{with} \quad N_z f = zf.$$

Then, N_z defines a normal operator on $L^2(\mu)$. Furthermore,

$$p_n(\mu, z) = \kappa_n(\mu) \det(z - \pi_n N_z \pi_n),$$

where π_n is the projection onto onto $\mathbb{P}_{n-1}$.

Theorem (B. Simon, Duke Math. J., 2009)

Let

$$N(\mu) := \sup\{|z| : z \in S_\mu\}.$$

Then, for any $k \in \mathbb{N}$,

$$\pi_n N_z^k \pi_n - (\pi_n N_z \pi_n)^k,$$

is an operator of rank at most k and norm at most $2N(\mu)^k$.

Shift Operator on $L^2(\mu)$

Let ν_n denote the normalized counting measure of zeros of p_n and let μ_n be defined by $d\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} |p_n(\mu, z)|^2 d\mu(z)$.

Theorem (B. Simon, Duke Math. J., 2009)

$$\frac{1}{n}\operatorname{Tr}(\pi_n N_z \pi_n)^k = \int z^k d\nu_n.$$

$$\frac{1}{n}\operatorname{Tr}(\pi_n N_z^k \pi_n) = \int z^k d\mu_n.$$

Thus, from the previous theorem, for any $k = 0, 1, 2, \ldots,$

$$\left| \int z^k d\nu_{p_n} - \int z^k d\mu_n \right| \leq \frac{2kN^k(\mu)}{n}.$$

Furthermore, if K is a compact set containing the supports of all ν_n and μ, such that $\{z_k\}_{k=0}^{\infty} \cup \{\overline{z}_k\}_{k=0}^{\infty}$ are $\|\cdot\|_\infty$-total in $\mathcal{C}(K)$, then for any subsequence $\{n_j\}$, $\boxed{\nu_{n_j} \xrightarrow{*} \nu}$ if and only if $\boxed{\mu_{n_j} \xrightarrow{*} \nu}$.

Krylov subspaces

Let $A \in \mathcal{L}(H)$ be a linear bounded operator acting on the complex Hilbert space H and let $\xi \in H$ be a non-zero vector. We denote by $H_n(A, \xi)$ the linear span of the vectors $\xi, A\xi, ..., A^{n-1}\xi$ and let π_n be the orthogonal projection of H onto $H_n(A, \xi)$. Let a_n denote the counting measure of the spectra of the *finite central truncation* $A_n = \pi_n A \pi_n$. Note that for any complex polynomial $p(z)$ it holds that

$$\int p(z) da_n(z) = \frac{\mathrm{Tr}(p(A_n))}{n}.$$

The orthogonal monic polynomials P_n in this case are defined as minimizers of the functional (semi-norm):

$$\|q\|^2_{A,\xi} = \|q(A)\xi\|^2, \quad q \in \mathbb{C}[z],$$

and the zeros of P_n (whenever P_n exists) coincide with the spectrum of A_n.

Theorem (Gustafsson & Putinar, Hyponormal Quantization of Planar Domains, Springer 2017)

Let $A, B \in \mathcal{L}(H)$ with $A - B$ of finite trace. Then, for every polynomial $p \in \mathbb{P}[z]$, we have

$$\lim_{n\to\infty} \frac{\text{Tr}(p(A_n)) - \text{Tr}(p(B_n))}{n} = 0.$$

Corollary

Let a_n, b_n denote the counting measures of the spectra of A_n and B_n, respectively. Then,

$$\lim_{n\to\infty} [\int \frac{da_n(\zeta)}{\zeta - z} - \int \frac{db_n(\zeta)}{\zeta - z}] = 0,$$

uniformly on compact subsets which are disjoint of the convex hull of $\sigma(A) \cup \sigma(B)$.

Conclusion

All the results in this section yield information about the analytic moments:

$$\lim_{n\to\infty} \int z^k d\nu_n = \int z^k d\nu, \quad k = 0, 1, 2, \ldots,$$

where ν is a known positive measure and $\{\nu_n\}$ are a sequence of positive measures (all supported on the same compact set K) we want to describe its limit points. Note that the measures being positive implies the same information for the anti-analytic moments:

$$\lim_{n\to\infty} \int \overline{z}^k d\nu_n = \int \overline{z}^k d\nu, \quad k = 1, 2, \ldots.$$

Conclusion

However, according to the complex Stone-Weierstrass theorem, in order to establish

$$\nu_n \xrightarrow{*} \nu,$$

we need the limits of all the complex moments

$$\lim_{n\to\infty} \int z^k \overline{z}^j d\nu_n = \int z^k \overline{z}^j d\nu, \quad k, j = 0, 1, 2, \dots,$$

unless K is of a special form, where the analytic moments constitute sufficient information.