OPSFA 2019
July 22-26

15th International Symposium on
Orthogonal Polynomials, Special Functions & and Applications
Research Institute for Symbolic Computation & Johann Radon Institute for Computational and Applied Mathematics • Hagenberg, Austria

Rational Approximation & Sobolev Orthogonal Polynomials

(Markov's Theorem and its extension to Sobolev-type orthogonality)

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† Research partially supported by Ministry of Science, Innovation and Universities of Spain, under grant PGC2018-096504-B-C33.

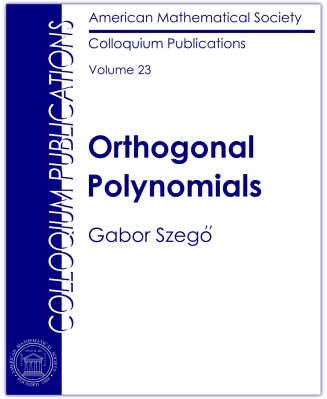
‡ Supported by the Research Fellowship Program, Ministry of Economy and Competitiveness of Spain, under grant MTM2015-65888-C4-2-P.

∗ Research partially supported by Fondo Nacional de Innovación y Desarrollo Científico y Tecnológico (FONDOCYT), Dominican Republic, under grant 2015-1D2-164.

Historically, the ***orthogonal polynomials*** $\{p_n(x)\}$ ***originated in the theory of continued fractions***. *This relationship is of great importance and is one of the possible* ***starting points*** *of the treatment* ***of orthogonal polynomials***.

Gabor Szegő

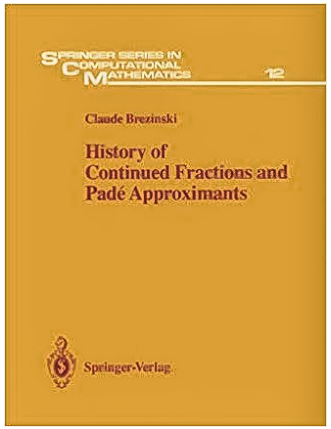
Orthogonal Polynomials (§3.5)



Nowadays, the theory of orthogonal polynomials is usually presented without any reference to continued fractions. In fact historically, the theory of orthogonal polynomials originated from certain types of continued fractions. *Special cases of such continued fractions were studied by C.F. Gauss, C.G. Jacobi, E.B. Christoffel and G. F. Mehler, while more general aspects were given by P.L. Chebyshev, H.E. Heine and* ***A.A. Markov***.

Claude Brezinski

History of Continued Fractions and Padé Approximants (§5.2.3)



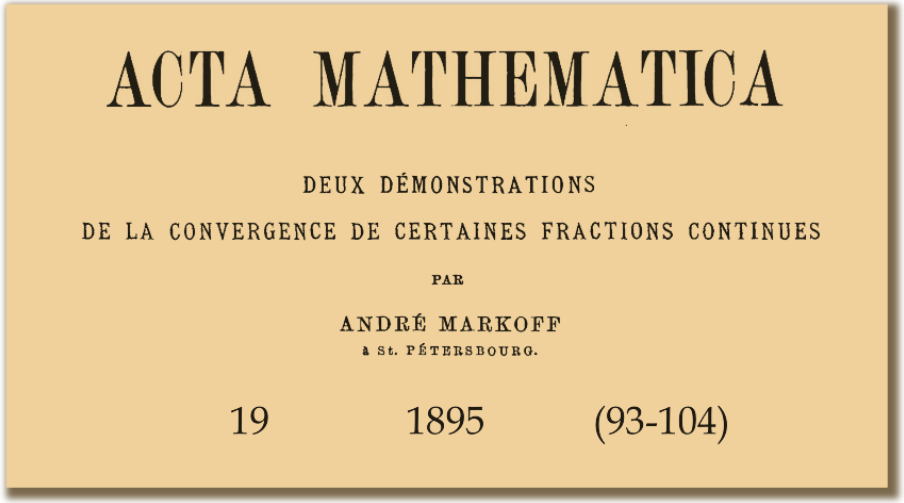
Markov's Theorem

- μ finite positive Borel measure with support $[-1,1]$,
- P_n nth monic orthogonal polynomials w.r.t. μ.
- $P_n^{[1]}(z) = \displaystyle\int_{-1}^{1} \frac{P_n(z)-P_n(t)}{z-t} d\mu(t)$ the associated polynomial.

Then

$$\frac{P_n^{[1]}(z)}{P_n(z)} \underset{n\to\infty}{\rightrightarrows} \widehat{\mu}(z) = \int_{-1}^{1} \frac{d\mu(x)}{z-x}, \quad K \subset \Omega_\infty = \overline{\mathbb{C}} \setminus [-1,1].$$

A.A. **Markov**
(1856-1922)



ACTA MATHEMATICA

DEUX DÉMONSTRATIONS
DE LA CONVERGENCE DE CERTAINES FRACTIONS CONTINUES

PAR

ANDRÉ MARKOFF
à St. PÉTERSBOURG.

19 1895 (93-104)

Complement of Markov's theorem

$$\limsup_{n\to\infty} \left\| \widehat{\mu} - \frac{P_n^{[1]}}{P_n} \right\|_K^{\frac{1}{2n}} \leq \kappa < 1,$$

Sobolev orthogonality vs. standard orthogonality

$$\langle p, q\rangle_S = \int p(x)q(x)d\mu_0(x) + \sum_{k=1}^{N} \int p^{(k)}(x)q^{(k)}(x)d\mu_k(x), \quad N > 0. \tag{1}$$

(Sobolev-type or discrete if $\{\mu_k\}_{k=1}^{N}$ are discrete measures)

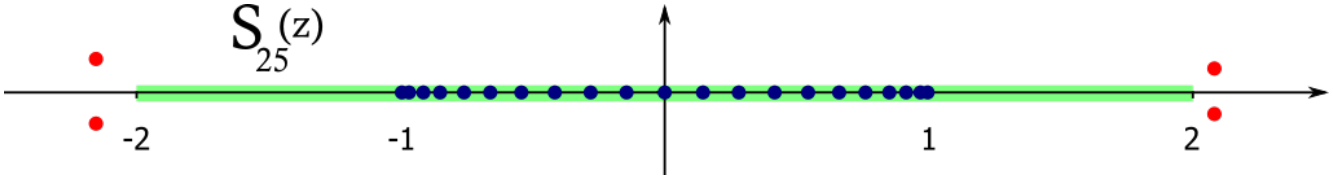
Let $S_n \perp \langle\cdot,\cdot\rangle_S$ monic, $\deg(S_n) = n$.

1.- $\langle xp, q\rangle_S \neq \langle p, xq\rangle_S$ (non-standard), except trivial examples. Hence, $\{S_n\}$ does not satisfy a 3TRR*†

2.- The matrix of moments is not a Hankel matrix‡.

3.- The zeros of S_n are not necessarily on Co $\left(\bigcup_{k=1}^{N} \text{supp}\mu_k\right)$. For example:

$$\langle p, q\rangle_S = \int_{-1}^{1} p(x)q(x)dx + p''(-2)q''(-2) + p'''(-2)q'''(-2) + p'(2)q'(2) + p''(2)q''(2).$$



* A. Durán, **A generalization of Favard's Theorem for polynomials satisfying a recurrence relation**, J. Approx. Theory, **74** (1993), 83-109.

† W. D. Evans, L. Littlejohn, F. Marcellán, C. Markett, A. Ronveaux, **On recurrence relations for Sobolev orthogonal polynomials**, SIAM J. Math. Anal. , **26**(1995), 446–467.

‡ D. Barrios, G. López, H. Pijeira, **The moment problem for a Sobolev inner product**, J. Approx. Theory, **100** (1999) 364-380.

Sobolev-type inner product

$$\langle f, g\rangle_S = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + \sum_{j=1}^{N}\sum_{i=0}^{d_j} \eta_{j,i} f^{(i)}(c_j) g^{(i)}(c_j); \tag{2}$$

where μ is a finite positive Borel measure supported on $[-1,1]$, $N \geq 0$, $\eta_{j,i} \geq 0$, $\eta_{j,d_j} > 0$, $c_j \in \mathbb{R} \setminus [-1,1]$ and $d_j \in \mathbb{Z}_+$. S_n **the nth monic orthogonal polynomial w.r.t.** (2)**.**

- Q_n **the nth monic orthogonal polynomial w.r.t.** $\langle \cdot, \cdot \rangle_{\mu_\rho}$**,** where

$$\langle f, g\rangle_{\mu_\rho} = \int_{-1}^{1} f(x)\; g(x)\; d\mu_\rho(x), \tag{3}$$

$$\rho(z) = \prod_{c_j<-1} (z-c_j)^{d_j+1} \prod_{c_j>1} (c_j - z)^{d_j+1} \text{ and } d\mu_\rho(x) = \rho(x) d\mu(x).$$

- If $n > d = \sum_{j=1}^{N} (d_j + 1)$, $\{S_n\}$ satisfies the next quasi-orthogonality relations w.r.t. $d\mu_\rho$

$$\langle S_n, f\rangle_{\mu_\rho} = \langle S_n, \rho f\rangle_\mu = \int_{-1}^{1} S_n(x) f(x) \rho\, d\mu(x) = \langle S_n, \rho\, f\rangle = 0, \quad \text{for all } f \in \mathcal{P}_{n-d-1}.$$

- If $n > d$, then S_n has at least $(n-d)$ changes of sign in $(-1,1)$ (note that $d \leq N$).

Definition 3.1 (kth **associated polynomial** to S_n, $k \in \mathbb{N}$ and $n \geq 1$.)

$$S_n^{[k]}(z) = \int_{-1}^{1} \frac{S_{n+k}(z) - S_{n+k}(x)}{z - x} \, Q_{k-1}(x) \, d\mu_\rho(x)$$

$$S_n^{[0]}(z) = S_n(z).$$

Properties

1.▶ Let $S_n^{[k]}$ be the kth associated polynomial. Then $S_n^{[k]}$ is a polynomial of degree n and leading coefficient equal to $\|Q_{k-1}\|_{\mu_\rho}^2$.

2.▶ For $n \geq 2d - 1$, the sequences $\{S_n^{[k]}\}_{n=0}^{\infty}$ satisfy the following $2d + 1$ term recurrence relation

$$\rho(z) S_n^{[k]}(z) = \sum_{j=n-d}^{n+d} \mathfrak{a}_{n+k,j+k} \, S_j^{[k]}(z), \quad \text{where} \quad \mathfrak{a}_{n+k,j+k} = \frac{\langle S_{n+k}, \rho S_{j+k} \rangle}{\langle S_{j+k}, S_{j+k} \rangle}. \tag{4}$$

3.▶ For $n \geq d - 1$, the sequences $\{S_n^{[k]}\}_{n=0}^{\infty}$, for $k \geq 2$, hold the following relation

$$S_n^{[k]}(z) = (z - \beta_{k-2}) S_{n+1}^{[k-1]}(z) - \alpha_{k-2}^2 S_{n+2}^{[k-2]}(z). \tag{5}$$

Sobolev-type inner product **sequentially-ordered**

Definition 4.1

Let $\{(r_j, \nu_j)\}_{j=1}^{M} \subset \mathbb{R} \times \mathbb{Z}_+$ be a finite sequence of M ordered pairs and $A \subset \mathbb{R}$. We say that $\{(r_j, \nu_j)\}_{j=1}^{M}$ is *sequentially-ordered with respect to* A, if

1.- $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_M$.

2.-(*) $r_k \notin \mathbf{C}_h\left(A \cup \{r_1, r_2, \ldots, r_{k-1}\}\right)$ for $k = 1, 2, \ldots, M$.

If $A = \emptyset$, we say that $\{(r_j, \nu_j)\}_{j=1}^{M}$ is *sequentially-ordered* for brevity.

Definition 4.2

The inner product (2) is **sequentially-ordered**, if the set of ordered pairs

$$\{(c_j, i) : 1 \leq j \leq N, 0 \leq i \leq d_j \text{ and } \eta_{j,i} > 0\}$$

may be arranged as a finite sequence of ordered pairs which is sequentially ordered with respect to $(-1, 1)$.

From 2.-(*), the coefficient η_{j,d_j} is the only coefficient $\eta_{j,i}$ $(i = 0, 1, \ldots, d_j)$ different from zero, for each $j = 1, 2, \ldots, N$. Hence, the inner product (2) takes the form

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, d\mu(x) + \sum_{j=1}^{N} \eta_{j,d_j} f^{(d_j)}(c_j) g^{(d_j)}(c_j). \tag{6}$$

▶ Examples

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + f'(2)g'(2) + f''(5)g''(5) + f'''(9)g'''(9).$$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + f'(2)g'(2) + f''(-2)g''(-2) + f''(-3)g''(-3).$$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + f'(-2)g'(-2) + f''(-2)g''(-2) + f'''(-2)g'''(-2).$$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + f'(9)g'(9) + f''(5)g''(5) + f'''(2)g'''(2).$$

■ sequentially-ordered ■ non sequentially-ordered

Theorem 4.3

If (6) *is a sequentially-ordered, then* S_n *has at least* $n - N$ *changes of sign on* $(-1, 1)$.

Theorem 4.4

If (6) *is sequentially-ordered and* $\mu \in \mathbf{M}(0,1)$ *(the Nevai class). Then*

$$\frac{S_n(z)}{P_n(z)} \underset{n\to\infty}{\Longrightarrow} \prod_{j=1}^{N} \frac{(\varphi(z) - \varphi(c_j))^2}{2\varphi(z)\,(z - c_j)}, \quad K \subset \overline{\Omega}, \tag{7}$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$, *with* $\sqrt{z^2 - 1} > 0$ *for* $z > 1$.

G. López Lagomasino, F. Marcellán, and W. Van Assche, ***Relative asymptotics for orthogonal polynomials with respect to a discrete Sobolev inner product****, Constr. Approx.,* ***11*** *(1995), 107-137.*

Corollary 4.3-4.4

1.- For all sufficiently large n, each sufficiently small neighborhood of c_j; $j = 1, \cdots, N$; contains exactly one zero of S_n and the remaining $n - N$ zeros lie on $(-1, 1)$.

2.- For all sufficiently large n, the **zeros** of S_n are **real and simples**.

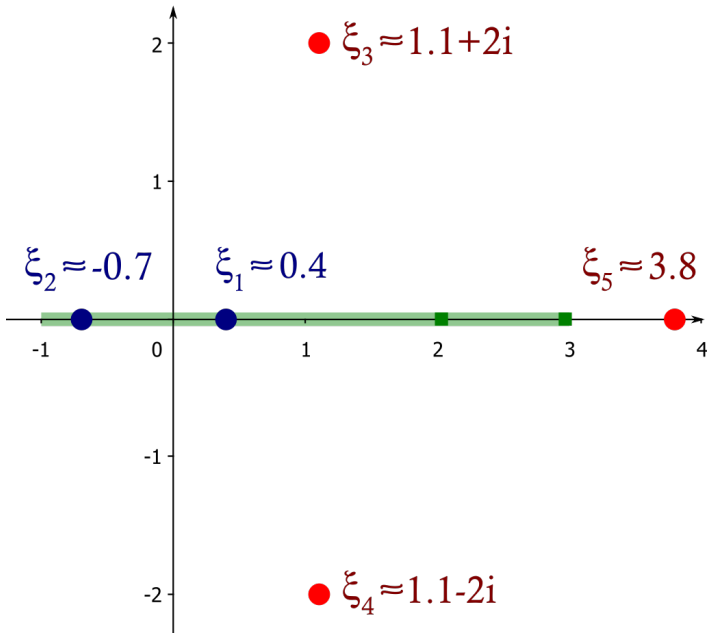
3.- The set of zeros of $\{S_n\}_{n=1}^{\infty}$ is **uniformly bounded**.

Non-sequentially-ordered Sobolev-type inner product

Example 1.

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + f''(2)g''(2) + f'(3)g'(3).$$

$$S_5(x) = x^5 + \frac{11282625}{1995289}x^4 + \frac{202236410}{1795760}x^3 + \frac{28506900}{1995289}x^2 - \frac{438413755}{41901069}x - \frac{11758825}{1995289},$$ whose zeros are

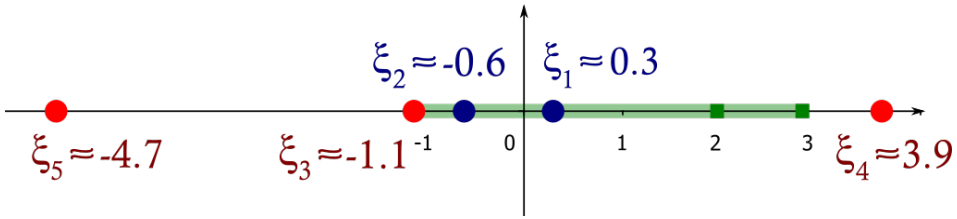


Example 2.

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x)\,dx + f'(3)g'(3) + f''(2)g''(2).$$

$$S_5(x) = x^5 + \frac{57943145}{27312164}x^4 - \frac{242237045}{13656082}x^3 - \frac{522277585}{20484123}x^2 - \frac{53214815}{40968246}x + \frac{220912645}{52141404},$$

whose zeros are



Markov's theorem for Sobolev orthogonality

Extended Markov's Theorem

Let (6) be a discrete Sobolev inner product sequentially-ordered with $\mu \in \mathbf{M}(0,1)$. Then, for $k \in \mathbb{N}$,

$$R_n^{[k]} = \frac{S_n^{[k]}(z)}{S_{n+k}(z)} \underset{n\to\infty}{\Longrightarrow} \widehat{\mu}_k(z) = \int_{-1}^{1} \frac{Q_{k-1}(x)}{z-x} d\mu_\rho(x),$$

$$K \subset \overline{\mathbb{C}} \setminus \left([-1,1] \cup \{c_1, c_2, \cdots, c_N\}\right).$$

We call $\widehat{\mu}_k$ the *kth Markov-type function associated to* μ_ρ.

Corollary (estimate for the degree of convergence)

$$\limsup_n \left\| \widehat{\mu}_k - R_n^{[k]} \right\|_K^{1/2n} \leq \|\varphi^{-1}\|_K < 1, \quad \text{where } \|f\|_k = \sup_{z\in K} |f|.$$

Prologue of the proof and auxiliary lemmas

- Let $\{\xi_{n,i}\}_{i=1}^{n-N}$ be the $n-N$ simple zeros of S_n on $(-1,1)$ for $(n \geq d + d_N)$ and let $\{\xi_{n,n-N+i}\}_{i=1}^{N}$ be the remaining N zeros of S_n.
- $S_n(x) = S_{n,1}(x)\, S_{n,2}(x)$, where $S_{n,1}(x) = \prod_{i=1}^{n-N}(x - \xi_{n,i})$ and $S_{n,2}(x) = \prod_{i=1}^{N}(x - \xi_{n,n-N+i})$.
- $S_{n,2}^{+}(x) = (-1)^{\nu} S_{n,2}(x)$ where ν the number of c_j greater than 1.
- If n is sufficiently large, $S_{n+k,2}^{+}(x)\ \rho(x) > 0$ for all $x \in [-1,1]$.
- For each $k \in \mathbb{N}$,

$$R_{n,1}^{[k]}(z) = \frac{S_{n,1}^{[k]}(z)}{S_{n+k,1}(z)}. \qquad S_{n,1}^{[k]}(z) = \int_{-1}^{1} \frac{S_{n+k,1}(z) - S_{n+k,1}(x)}{z-x}\, Q_{k-1}(x)\, d\mu_{\rho,n}(x).$$

$$d\mu_{\rho,n}(x) = S_{n+k,2}^{+}(x)\ \rho(x)\ d\mu(x). \qquad \widehat{\mu}_{k,n}(z) = \int_{-1}^{1} \frac{Q_{k-1}(x)}{z-x} d\mu_{\rho,n}(x).$$

Lemma 1 (Gauss-Jacobi type quadrature formula)

Let S_n and $\{\xi_{n,i}\}_{i=1}^{n-N}$ as above. If $n > d$ then for every polynomial T with $\deg(T) \leq 2n - d - N - 1$

$$\int_{-1}^{1} T(x) S_{n,2}^{+}(x) d\mu_\rho(x) = \sum_{i=1}^{n-N} \lambda_{n,i}\, S_{n,2}^{+}(\xi_{n,i})\, T(\xi_{n,i}),$$

$$\text{where} \quad \lambda_{n,i} = \int_{-1}^{1} \frac{S_n(x)}{S_n'(\xi_{n,i})(x - \xi_{n,i})} d\mu_\rho(x) \ \textbf{(Christoffel-type number)}.$$

Moreover, the number of positive coefficients $\lambda_{n,i}$ is greater than or equal to $\left(n - \frac{d+N}{2}\right)$.

Lemma 2 (estimating the remainder)

$$\widehat{\mu}_{k,n}(z) - R_{n,1}^{[k]}(z) = S_{n+k,2}^{+}(z)\left(\widehat{\mu}_k(z) - R_n^{[k]}(z)\right) = \mathcal{O}\left(\frac{1}{z^{2(n+1)+k-d-N}}\right).$$

Lemma 3 (Fundamental Lemma)

- For n sufficiently large $R_{n,1}^{[k]}(z) = \displaystyle\sum_{j=1}^{n+k-N} \frac{S_{n+k,2}^{+}(\xi_{n+k,j})\ \lambda_{n+k,j}}{(z - \xi_{n+k,j})}$.
- **Furthermore, $\{R_{n,1}^{[k]}\}$ is uniformly bounded on each compact subset $K \subset \Omega$.**

Outline of the proof of Extended Markov's Theorem

- $K \subset \Omega_\infty$ compact and ℓ_τ the level curve $\ell_\tau = \{z \in \mathbb{C} : |\varphi(z)| = \tau\}$, where $\tau > 1$.
- Since $\varphi(K)$ is a compact set we can take τ sufficiently close to 1 such that $1 < \tau < \min |\varphi(K)|$.
- **From Lemma 3 the sequences $\{\widehat{\mu}_{k,n}\}$ and $\{R_{n,1}^{[k]}\}$ are uniformly bounded on K.** Then, there exist a constant C_τ, independent of n, such that for all $z \in \ell_\tau$

$$\left| \left(\widehat{\mu}_{k,n}(z) - R_{n,1}^{[k]}(z) \right) \varphi^{2(n+1)+k-d-N}(z) \right| \leq C_\tau \; \tau^{2(n+1)+k-d-N}. \tag{8}$$

- From Lemma 2 we have

$$\left(\left(\widehat{\mu}_{k,n}(z) - R_{n,1}^{[k]}(z) \right) \varphi^{2(n+1)+k-d-N}(z) \right) \in \mathbb{H}(\Omega_\infty).$$

- Now, **form the maximum modulus principle the bound** (8) **also holds on K.**

$$\text{Therefore} \quad \sup_{z \in K} \left| \widehat{\mu}_{k,n}(z) - R_{n,1}^{[k]}(z) \right| \leq C_\tau \left(\frac{\tau}{\min |\varphi(K)|} \right)^{2(n+1)+k-d-N} \underset{n\to\infty}{\longrightarrow} 0,$$

which is equivalent to $R_{n,1}^{[k]}(z) \underset{n\to\infty}{\rightrightarrows} \widehat{\mu}_{k,n}(z) \quad K \subset \Omega_\infty$.

- From Corollary 4.3-4.4, there exist a constant $C_2 > 0$, independent of n, such that $C_2 \leq |S_{n+k,2}(z)|$ for all $z \in K$. Therefore, taking into account Lemma 2 we get

$$\sup_{z \in K} \left| \widehat{\mu}_k(z) - R_n^{[k]}(z) \right| \leq \frac{C_\tau}{C_2} \left(\frac{\tau}{\min |\varphi(K)|} \right)^{2(n+1)+k-d-N} \underset{n\to\infty}{\longrightarrow} 0.$$

□

Thank You
for your attention