Some q-series conjectures related to Rogers-Ramanujan type identities of Kanade and Russell

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The Kanade-Russell Conjectures

Between two articles (2015 and 2019), Kanade and Russell conjectured about 20 identities of the Rogers-Ramanujan type.

The conjectures are phrased both in terms of "number of partitions with gap conditions" $=$ "number of partitions with congruence condititions" and "multi-sum $=$ infinite product". In the case of their so-called modulo 9 conjectures, the multi-sums were given by Kurșungöz.

Kanade and Russell released a MAPLE package, `IdentityFinder`, to conduct a brute force search for conjectures of certain shapes of "gap conditions" $=$ "congruence condititions".

The Kanade-Russell Conjectures

Two examples:

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+4i+6j+12k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = \frac{1}{(q^4,q^5,q^6,q^7,q^8;q^{12})_\infty},$$

$$\sum_{i,j,k\geq 0} \frac{q^{\frac{1}{2}(i+2j+3k)(i+2j+3k-1)+j^2+i+2j+4k}}{(q;q)_i(q^2;q^2)_j(q^3;q^3)_k} = \frac{1}{(q;q^3)_\infty(q^3,q^6,q^{11};q^{12})_\infty},$$

where

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j),\ (a_1,\ldots,a_k;q)_n := (a_1;q)_n\cdots(a_k;q)_n.$$

Theorem (Bringmann, JS, Malhburg (2019))

The above two conjectures (and five of a similar shape) are true.

Rogers-Ramanujan Identities

These conjectures fit into the framework of Rogers-Ramanujan type identities. The original Rogers-Ramanujan identities (first discovered and proved by Rogers, then later rediscovered by Ramanujan), stated in sum=product form, are

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Multi-sums are part of this framework, with the most famous being the Andrews-Gordon identities:

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_j+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}$$
$$= \frac{(q^j;q^{2k+1})_\infty(q^{2k+1-j};q^{2k+1})_\infty(q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty},$$

where $1\leq j\leq k$, $k\geq 2$, and $N_\ell := \sum_{i=\ell}^{k-1} n_i$.

A Minimal Proof Sketch

Of the seven conjectures we proved, our method was to reduce the triple sums to single sums, by solving q-difference equations, and then proving the "single-sum"="product" with known q-series identities.
With

$$H(x) := \sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+4i+6j+12k} x^{i+2j+3k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k},$$

we have

$$\begin{aligned} H(x) = (1+xq^4)H(xq^2) + xq^5(1+xq^3)H(xq^4) \\ + x^2q^{12}(1-xq^4)H(xq^6). \end{aligned}$$

Recurrences of this type can be detected numerically.

Proof Sketch

For this we view the coefficients, such as $1 + xq^4$, on the left hand-side as polynomials in x and q, with the coefficients as unknowns. We then expand both sides and solve the associated linear system of equations (finitely many unknowns and infinitely many equations).

The recurrence are proved by solving a different linear system that comes from shifts and rearrangements of the triple sum. It is also possible to prove these recurrences combinatorially.

Ablinger and Uncu have recently automated this process with the Mathematica package `qFunctions` in tandem with Schneider's `Sigma` package.

Proof Sketch

We have recurrences for all of the Kanade-Russell conjectures, but not all are "useful". The hard part is getting "useful" recurrences.

Another conjecture is

$$\sum_{i,j,k,\ell\geq 0} \frac{(-1)^{\ell} q^{\frac{(i+2j+3k+4\ell)(i+2j+3k+4\ell-1)}{2}+2\ell^2+i+3j+6k+6\ell}}{(q;q)_i\,(q^2;q^2)_j\,(q^3;q^3)_k\,(q^4;q^4)_\ell}$$
$$= \frac{1}{(q,q^3,q^4,q^6,q^8,q^9,q^{11};q^{12})_\infty}.$$

Setting $G(x)$ to be the sum with an additional $x^{i+2j+3k+4\ell}$ yields

$$\begin{aligned} G(x) = {} & \left(1+xq-xq^2\right) G(xq) + xq^2\left(1+2xq^2-xq^3\right) G\left(xq^2\right) \\ & + x^2q^5\left(1-xq^2+2xq^3\right) G\left(xq^3\right) \\ & + x^3q^9\left(1-xq^3+xq^4\right) G\left(xq^4\right) - x^5q^{18} G\left(xq^5\right). \end{aligned}$$

Proof Sketch

Why is

$$H(x) = (1 + xq^4)H(xq^2) + xq^5(1 + xq^3)H(xq^4) + x^2q^{12}(1 - xq^4)H(xq^6)$$

useful?

Proposition (Bringmann, JS, Mahlburg)

Suppose that $A(x) = \sum_{n\geq 0} \alpha_n x^n$ has radius of convergence greater than 1, $A(0) = 1$, and

$$A(x) = (1 + xq^{a+2})A(xq^2) + xq^{a+b}(1 + xq^b)A(xq^4) + x^2q^{2a+2b+2}(1 - xq^4)A(xq^6).$$

Then $A(1) = (q^{2b+a-2}; q^4)_\infty \sum_{n\geq 0} \frac{(q^{3b-6}; q^6)_n q^{n^2+(a+1)n}}{(q^{b-2}, q^2; q^2)_n (q^{2b+a-2}; q^4)_n}.$

Proof Sketch

We prove this proposition by noting q-difference equations are equivalent to recurrences for the coefficients of the power series. We take turns making a substitution that reduces the q-difference equation and one that reduces the recurrence of the coefficients. We then have to track this all the way back to $A(1)$.

$$A(x) \Longrightarrow \qquad B(x) := A(x)/(x;q^2), \quad \sum \beta_n x^n := B(x),$$

$$\beta_n \Longrightarrow \qquad \gamma_n := (*)\beta_n, \quad C(x) := \sum \gamma_n x^n,$$

$$C(x) \Longrightarrow \qquad D(x) := (**)C(x), \quad \sum \delta_n x^n := D(x),$$

$$\delta_n \Longrightarrow \qquad \epsilon := (***)\delta_n, \quad E(x) := \sum \epsilon_n x^n.$$

Proof Sketch

The resulting $E(x)$ is

$$E(x) = \frac{(xq^{b-2};q^2)_\infty (x^2q^{2b+a-2};q^4)_\infty}{(x^3q^{3b-6};q^6)_\infty}.$$

There is no guarantee this process yields any useful information for $A(x)$. A substitution in the q-difference equation fundamentally changes the coefficients, and a substitution in the recurrence for the coefficients fundamentally changes the underlying function.

While we did this by hand, Ablinger and Uncu's package can guide a user through these substitutions.

Proof Sketch

This yields

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+4i+6j+12k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = (q^6;q^4)_\infty \sum_{n\geq 0} \frac{(q^3;q^6)_n q^{n^2+3n}}{(q,q^2;q^2)_n(q^6;q^4)_n}.$$

After some rearrangements, the original conjecture is equivalent to a known Rogers-Ramanujan type identity of McLaughlin and Sills.

This is basically what happens with the other six conjectures that we can prove. However, we also have "useful" q-difference equations that do not result in full proofs.

Partial Results

Another conjecture of Kanade and Russell is

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+3j+3k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = \frac{1}{(q;q^4)_\infty(q^4,q^{11};q^{12})_\infty}.$$

We can reduce this conjecture to

$$\sum_{n\geq 0} \frac{(q^3;q^6)_n q^{n^2}}{(q,q^2;q^2)_n(q^3;q^4)_n} = \frac{1}{(q;q^2)_\infty(q^4,q^{11};q^{12})_\infty}.$$

This looks fairly simple, but we have been unable to solve it. The standard list of identities from Gasper and Rahman does not appear to help. Bailey's Lemma does not appear to help. Other tricks such as taking a finite version of the series does not appear to help.

Partial Results

The single sum can be written as a $_2\phi_1$, and a $_2\phi_1 =$ *product* identity should be easy!

Let $\omega = e^{\frac{2\pi i}{3}}$,

$$\sum_{n\geq 0} \frac{(q^3;q^6)_n q^{n^2}}{(q,q^2;q^2)_n (q^3;q^4)_n} = \sum_{n\geq 0} \frac{(q,\omega q,\omega^{-1}q;q^2)_n q^{n^2}}{(q,q^2,q^{\frac{3}{2}},-q^{\frac{3}{2}};q^2)_n}$$

$$= \lim_{a\to\infty} \sum_{n\geq 0} \frac{(a,\omega q,\omega^{-1}q;q^2)_n \left(-\frac{q}{a}\right)^n}{(q^{\frac{3}{2}},-q^{\frac{3}{2}},q^2;q^2)_n}$$

$$= \frac{(-q,\omega q;q^2)_\infty}{(q^3;q^4)} \sum_{n\geq 0} \frac{(\omega^{-1}q^{\frac{1}{2}},-\omega^{-1}q^{\frac{1}{2}};q^2)_n \omega^n q^n}{(-q,q^2;q^2)_n},$$

where the last line is by Hall's $_3\phi_2$ transformation (equation III.10 in the appendix of Gasper and Rahman).

Partial Results

Is this ${}_2\phi_1$ a useful representation to work with, or is it a red herring?

Is the single sum

$$\sum_{n\geq 0} \frac{(q^3;q^6)_n q^{n^2}}{(q,q^2;q^2)_n(q^3;q^4)_n}$$

actually a better object to work with than the triple sum? It does at least lend credibility to the conjecture and makes the conjecture easier to test. In particular, my laptop can verify this form of the conjecture up to q^{3000} in several minutes.

It feels likely that one or both of these single sums are the way to go, but we are missing some small trick to proceed. At this point, it doesn't look like a hard problem, but we have been unable to solve it.

Partial Results

There are three other conjectures we can reduce to single sums in a similar manner.

This only accounts for about half of the total number of conjectures. These are the conjectures where we could find "useful" q-difference equations. What about the others?

One of the modulo 9 conjectures is:

$$\sum_{i,j\geq 0} \frac{q^{i^2+3j^2+3ij}}{(q;q)_i(q^3;q^3)_j} = \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}.$$

This form is due to Kurşungöz. He also gave the double sum with an extra parameter to keep track of the number of parts in the appropriate combinatorial partition interpretation.

Partial Results

Here the series is:

$$A(x) := \sum_{i,j\geq 0} \frac{q^{i^2+3j^2+3ij}}{(q;q)_i(q^3;q^3)_j} x^{i+2j}.$$

This form yields the q-difference equation:

$$\begin{aligned}
A(x) = {} & (1 + xq + xq^2 + xq^3 + x^2q^3 + x^2q^6 - x^2q^7)A(xq^3), \\
& + x^2q^4(1 + xq^5 - x^2q^{11} + x^2q^{12} + x^2q^{15})A(xq^6), \\
& - x^5q^{24}(1 + q^2 + q^3 + xq^{10})A(xq^9) \\
& - x^7q^{46}A(xq^{12}).
\end{aligned}$$

However, slight variants give better recurrences. At this point, we are only interested in $x = 1$, as none of the conjectures yet have an "x" in the product side.

Partial Results

If we instead use

$$A_1(x) := \sum_{i,j\geq 0} \frac{q^{i^2+3j^2+3ij}}{(q;q)_i(q^3;q^3)_j} x^{i+3j},$$

then

$$\begin{aligned} A_1(x) = {} & (1-x)\, A_1(xq) + x\left(1+q+x^2q^3\right) A_1\left(xq^2\right) \\ & + x^2q^2 A_1\left(xq^3\right). \end{aligned}$$

This still looks difficult to solve, since it is not clear what substitution should be made to reduce the q-difference equation, and the associated recurrence for the coefficients is order 3. However, this q-difference equations is the simplest among those that are "useless", so perhaps they are not completely "useless".

This kind of trickery was necessary to prove some of the conjectures. The q-difference equation and recurrences might yield proofs of more conjectures.

Thank You for Your Attention

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