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A survey on bi-orthogonal polynomials and functions

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Abstract

The theory of orthogonal polynomials is well established and detailed, covering a wide field of interesting results, as in particular for solving certain differential equations. On the other side the concepts and the related formalism of the theory of bi-orthogonal polynomials is less developed and much more limited. By starting from the orthogonality properties satisfied from the ordinary and generalized Hermite polynomials is possible to derive a further family (known in literature) of these kind of polynomials which are bi-orthogonal with their adjoint. This aspect allow us to introduce functions recognized as bi-orthogonal and investigate generalizations of families of orthogonal polynomials.

Presentation approach

The topic of bi-ortogonality will be treated using the formalism and the operational properties satisfied by different classes of polynomials recognizable as generalized Hermite polynomials. The consolidated approach to the study of the characteristics of orthogonality and, although less developed, the one related to the concept of bi-ortogonality, will be reread on the different formalisms that can be obtained from the various relations deducible from the structure of the different polynomials and the related functions attributable to the family of Hermite polynomials.

The fulcrum of the discussion is based on the two-dimensional extension of Hermite polynomials

$$H_n(x) \to H_{m,n}(x,y)$$

where we consider the variables as real and the indexes positive integers.

Presentation

- Orthogonality and bi-orthogonality
- Hermite polynomials and related functions
- Relevant relations involving bi-orthogonal Hermite functions
- Further investigations

Orthogonality and bi-orthogonality

By starting from the space $L_w^2(a,b)$ and by considering a real function $w(x)$ on the interval (a,b), non-negative, measurable and non-zero, we can define the inner product (if for every $\mathrm{f(x)} \in L_w^2(a,b)$, $f(x)^2 w(x)$ is Lebesgue integrable):

$$< \mathrm{f,g} > \stackrel{\text{def}}{=} (f,g)_w := \int_a^b f(x)g(x)w(x)dx$$

which define an Hilbert space. The functions f and g will be said orthogonal, if:

$$< f, g > = 0$$

and the function $w(x)$ is usually referred as the *weight function*.

Remark: for our purpose it is not necessary to introduce the concept of orthogonality through the Lebesgue-Stieltjes integral, by introducing a general measure $m(x)$ on $\mathbb{R}$.

By assuming that:

$$x^k \in L_w^2(a,b), \forall k \in \mathbb{N}$$

i.e.

the moment sequence $\mu_k, k = 0,1, \dots$ finite, $\mu_k := \int_a^b x^k w(x)dx, k = 0,1,2, \dots$

Under these hypothesis, we can construct the Hankel determinant which allows us to define the family of orthogonal polynomials by the following way:

$$\begin{cases} P_0(x) = \mu_0^{-1/2} \\ P_n(x) = \dfrac{1}{\sqrt{H_n H_{n+1}}} \begin{vmatrix} \mu_0 & \dots & \mu_n \\ \dots & \dots & \dots \\ 1 & \dots & x^n \end{vmatrix} \end{cases}$$

where:

$$H_{n+1} = \begin{vmatrix} \mu_0 & \dots & \mu_n \\ \dots & \dots & \dots \\ \mu_n & \dots & \mu_{2n} \end{vmatrix}$$

In a more direct way, a sequence of polynomials $\{P_n(x)\}_{n=0}^{+\infty}$ is said orthogonal if, for any $n, m \in \mathbb{N}$, we have:

(i) $P_n(x)$ is a polynomial of exactly degree n

(ii) $< P_n, P_m > = K_n \delta_{n,m}$

$K_n > 0$, $\delta_{n,m}$ denotes the Kronecker delta

the sequence is said orthonormal if:

(ii)' $< P_n, P_m > = \delta_{n,m}$

To introduce the bi-orthogonality, we need to define a totally positive kernel; that is a two-variable continuous real function $K(x, y)$ such that, for all:

$$x_1 < x_2 < \ \dots < x_m,\ y_1 < y_2 < \ \dots < y_m$$

it holds that:

$$det\left[K\left(x_i, y_j\right)\right]_{1 \le i,j \le m} > 0$$

By assuming the above condition and furthermore that the integrals involved are defined and finite, we can define an inner (bi-)product.

Let $p, q \in \mathcal{P}$, we said, nonsymmetric inner (bi-)product, the relation:

$$< p(x), q(x) >:= \iint_D \; p(x)q(y)K(x,y)dxdy$$

where $\mathcal{P}$ is the space of real polynomials, $D \neq \emptyset, D \subseteq \mathbb{R}^2$.

To better understand the nature of bi-orthogonality, we set:

Definition 1 Two sequences of polynomials $\{P_n(x)\}_{n=0}^{+\infty}$ and $\{Q_n(y)\}_{n=0}^{+\infty}$ are said to be bi-orthogonal polynomial sequences, if for any $n, m \in \mathbb{N}$:

(i) $P_n(x)$ and $Q_m(x)$ are polynomials of exactly degree n and m, respectively

(ii) $< P_n, Q_m > = K_n \delta_{n,m}, K_n > 0$

if $K_n = 1$, the sequences are said *bi-orthonormal*

Remark: in general, since the above definition:

$$< P_n, Q_m > \neq < Q_m, P_n >$$

This standard definition of bi-orthogonality is strictly based on the concept of orthogonality.

M. Bertola, M. Gekhtman, J. Szmigielski, *Cauchy Biorthogonal Polynomials*, 2010, Journal of Approximation Theory, 162, 832–867

In the next sections, we will describe the bi-orthogonality relations for a special class of Hermite polynomials which can be seen as a particular case of the previous definition, but from the other side, as a first step to generalize the concept itself in a different context.

Hermite polynomials and related functions

We start to introduce the ordinary one-variable Hermite polynomials and their orthogonal properties, by setting their explicit form:

$$H_n(x) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r x^{n-2r}}{r!\,(n-2r)!\,2^r}$$

and the related generating function, who reads:

$$\exp(xt - t^2) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x)$$

The ordinary Hermite polynomials are orthogonal on the interval $(-\infty, +\infty)$, with respect to the weight function:

$$e^{-\frac{x^2}{2}}$$

i.e.

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)e^{-\frac{x^2}{2}}dx = n!\sqrt{2\pi}\delta_{n,m}$$

The above relation suggest us to introduce a family of functions, based on the ordinary Hermite themselves in such a way as similar properties are derived, we set:

$$he_n(x) = \left(\frac{1}{\sqrt{2\pi}n!}\right)^{\frac{1}{2}} H_n(x)e^{-\frac{x^2}{4}}$$

which are orthonormal on the interval $(-\infty, +\infty)$:

$$\int_{-\infty}^{+\infty} he_n(x)he_m(x)dx = \delta_{n,m}$$

By exploiting the properties and the related formalism of the ordinary Hermite polynomials, it is possible to introduce the shift operators acting on the above Hermite functions:

$$\hat{a}_+ = \left(-\frac{d}{dx} + \frac{x}{2}\right), \hat{a}_- = \left(\frac{d}{dx} + \frac{x}{2}\right)$$

which gives:

$$\hat{a}_+ he_n(x) = \sqrt{n+1}\, he_{n+1}(x), \quad \hat{a}_- he_n(x) = \sqrt{n}\, he_{n-1}(x)$$

to obtain the following relation:

$$\hat{a}_+ \hat{a}_- he_n(x) = n he_n(x)$$

and in differential forms, reads:

$$\left[\frac{d^2}{dx^2} - \frac{x^2}{4} + \left(n + \frac{1}{2}\right)\right] he_n(x) = 0$$

by the definition of the one-variable Hermite functions $he_n(x)$. We can now generalize the above results obtained by using the ordinary one-variable Hermite polynomials and applying them to the two-variable Hermite polynomials family. We remind that the explicit forms of the generalized Hermite polynomials of type $H_n(x, y)$ reads:

$$H_n(x, y) = \sum_{r=0}^{[n/2]} \frac{n!}{r!(n-2r)!} (-y)^r (2x)^{n-2r}$$

We expect that it is also possible to define analogous Hermite functions of two variables that are orthogonal by using the expression of the generalized two-variable Hermite polynomials: we face the question starting directly

Definition 2 Let x and y be two real variables and $he_n(x)$ be the one-variable Hermite function. We define the two-variable Hermite function $he_n(x,y)$ given by the expression:

$$h_{ne}(x,y) = \sum_{r=0}^{[n/2]} \sqrt{\frac{n!}{r!\,(n-2r)!}}\, he_{n-2r}(x) he_r(y)$$

It is immediately to prove that the Hermite functions $he_n(x,y)$ are orthogonal on the interval $(-\infty,+\infty) \times (-\infty,+\infty)$:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} he_n(x,y) he_m(x,y) dx = \sum_{r=0}^{[n/2]} \frac{n!}{r!\,(n-2r)!}$$

It could be useful to observe that the term obtained in the above orthogonal relation can be read as a special case of the two-variable Hermite polynomials of the type $H_n'(x, y)$, that is:

$$H_n'\left(\frac{1}{2}, -1\right) = \sum_{r=0}^{[n/2]} \frac{n!}{r!(n-2r)!}$$

We can derive the generating function for the two-variable orthogonal Hermite functions $he_n(x, y)$ by using the structure and the identities of the Hermite polynomials:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)-\frac{1}{2}(y-t^2)} e^{\frac{x^2+y}{4}} = \sum_{n=0}^{+\infty} \frac{t^n}{\sqrt{n!}} he_n(x, y)$$

Since the two-variable Hermite polynomials satisfy many interesting identities it is possible to derive similar relations for the two-variable Hermite functions $he_n(x,y)$; it is remarkable to reminder the following differential relation:

$$x he_n(x,y) + (2y-1)\sqrt{n}\, he_{n-1}(x,y) - 2\sqrt{n(n-1)(n-2)}\, he_{n-3}(x,y) =$$

$$=\sqrt{n+1}\, he_{n+1}(x,y)$$

P. Appell and J. Kampè de Fèriet, *Fonctions hypergeometriques et hyperspheriques. Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926

H.M. Srivastava and H.L. Manocha, *A treatise on generating functions*, Wiley, New York, 1984

H.W. Gould and A.T. Hopper, *Operational formulas connected with two generalizations of Hermite Polynomials*, 1962, Duke Math. J., 51–62

Relevant relations involving bi-orthogonal Hermite functions

In the previous sections we have introduced the one-variable, one-index Hermite polynomials $H_n(x)$ and their generalization $H_n(x, y)$. It is possible to use the polynomials $H_n(x)$ to introduce a different class of Hermite polynomials with two indexes and two variables, which are a *vectorial extension:* this means that from an index acts on a one-dimensional variable, we will have a couple of indexes acting on a two-dimensional variable:

$$H_n(x) \to H_{m,n}(x, y)$$

Let the positive quadratic form:

$$q(x, y) = ax^2 + 2bxy + cy^2, a, c > 0, \Delta = ac - b^2 > 0$$

where a, b, c are real numbers and the associated matrix reads:

$$\widehat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \left|\widehat{M}\right| > 0$$

and by considering a vector $\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the space $\mathbb{R}^2$, it follows:

$$q(\underline{z}) = \underline{z}^t \widehat{M} \underline{z}$$

$$q(\underline{z}) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

Definition 3 We will called two-index, two-variable Hermite polynomials, indicate with the symbol $H_{m,n}(x,y)$, the polynomials defined by the following generating function:

$$e^{\underline{z}^t \hat{M} \underline{h} - \frac{1}{2} \underline{h}^t \hat{M} \underline{h}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x,y)$$

where:

$$\underline{z} = \binom{x}{y}, \underline{h} = \binom{t}{u} \in \mathbb{R}^2 \text{ with } t \neq u, (|t|, |u|) < +\infty$$

These polynomials are exploited in many fields of pure and applied mathematics, they are very useful in description of the quantum treatment of coupled harmonic oscillator.

Y.S. Kim and M.E. Noz, *Phase-space picture of quantum mechanics*, World Scientific, Singapore, 1991

By using the definition of the quadratic form, we can introduce the related adjoint class of these polynomials; by setting:

$$\bar{q}(\underline{z}) = \underline{z}^t \widehat{M}^{-1} \underline{z}$$

we have:

$$e^{\underline{v}^t \widehat{M}^{-1} \underline{k} - \frac{1}{2} \underline{k}^t \widehat{M}^{-1} \underline{k}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{r^m}{m!} \frac{s^n}{n!} G_{m,n}(x, y)$$

where:

$\underline{v} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, $\underline{k} = \begin{pmatrix} r \\ s \end{pmatrix} \in \mathbb{R}^2$ with $\underline{v} = \widehat{M}\underline{z}$, $\underline{k} = \widehat{M}\underline{h}$ and r$\neq s$, $(|r|, |s|) < +\infty$

The expression of the generating function defining the adjoint Hermite polynomials of two-index and two-variable $G_{m,n}(x,y)$, could be recast in the following form:

$$e^{\underline{z}^t\underline{k}-\frac{1}{2}\underline{k}^t\hat{M}^{-1}\underline{k}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{r^m}{m!}\frac{s^n}{n!}G_{m,n}(x,y)$$

The above introduced *vectorial* Hermite polynomials satisfy, with their adjoints, many and relevant properties. It is now interesting to explore the possibility to find similar Hermite functions as those defined in the previous section, in order to obtain an extension of the concepts and the related identities satisfied from the Hermite polynomials $H_{m,n}(x,y)$ and their adjoints $G_{m,n}(x,y)$.

The structure of the vectorial extension Hermite polynomials is based on the fact that a vector index acts on a vector variable or, that is the same, a couple of indexes act on a couple of variables. This suggests that we can not expect the same relation linking the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ and the related Hermite functions we are going to define, moreover the concept of orthogonality is not the same as the existing one for the one-index Hermite polynomials of type $H_n(x)$ and $H_n(x,y)$.

The two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and their adjoint $G_{m,n}(x,y)$ satisfy the following bi-orthogonality condition:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} H_{m,n}(x,y) G_{r,s}(x,y) e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} dx = \frac{2\pi}{\sqrt{\Delta}} m!\, n!\, \delta_{m,r} \delta_{n,s}$$

Remark: the proof is based on the definition of the generating functions and use the vectorial techniques of differentiation.

The weight function:

$$e^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}}$$

is easily recognized similar to the related weight function for the ordinary Hermite polynomials $H_n(x)$. We can use the above result to define functions based on the two-index, two-variable Hermite polynomials.

Definition 4 Let the Hermite polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ we call two-index, two-variable Hermite functions, be the functions defined in the following way:

$$\overline{H}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}H_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\widehat{M}\underline{z}}$$

$$\bar{G}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}G_{m,n}(x,y)e^{-\frac{1}{4}\underline{z}^t\widehat{M}\underline{z}}$$

which are obviously bi-orthonormal.

To emphasize the relevance of the bi-orthogonal Hermite functions, we see how to extend the differential relations showed for the polynomials $H_n(x)$.

We start to observe that the Hermite polynomials of type $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ solve the following partial differential equation:

$$\frac{\partial}{\partial\tau}S_{m,n}(x,y;\tau) = -\frac{1}{2}\begin{pmatrix}\partial_x & \partial_y\end{pmatrix}\widehat{M}^{-1}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix}S_{m,n}(x,y;\tau)$$

satisfying the condition at $\tau = 0$:

$$S_{m,n}(x,y;0) = \begin{cases}\xi^m\eta^n, when\ S_{m,n} = H_{m,n} \\ x^m y^n, when\ S_{m,n} = G_{m,n}\end{cases}$$

Remark: the proof is an immediate consequence of the properties satisfied by the polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$, deduced by their structure, using the similar well-known relations of the ordinary Hermite polynomials.

By defining the following shift operators:

$$\hat{a}_{+,0} = \frac{1}{2}(ax + by) - \frac{\partial}{\partial x}, \; \hat{a}_{-,0} = \frac{1}{\Delta}\left(c\,\frac{\partial}{\partial x} - b\,\frac{\partial}{\partial y}\right) + \frac{1}{2}x$$

$$\hat{a}_{0,+} = \frac{1}{2}(bx + cy) - \frac{\partial}{\partial y}, \; \hat{a}_{0,-} = -\frac{1}{\Delta}\left(b\,\frac{\partial}{\partial x} - a\,\frac{\partial}{\partial y}\right) + \frac{1}{2}y$$

where $\Delta = ac - b^2$ is the determinant of the quadratic form defined above

The above operators are free from any parameters, not presenting any index variable in their structure; therefore, different from the shift operators related to Hermite polynomials. The action of these operators could be summarize as follows:

$$\hat{a}_{+,0}\overline{H}_{m,n}(x,y) = \sqrt{m+1}\overline{H}_{m+1,n}(x,y)\ ,\ \hat{a}_{-,0}\overline{H}_{m,n}(x,y) = \sqrt{m}\overline{H}_{m-1,n}(x,y)$$

$$\hat{a}_{0,+}\overline{H}_{m,n}(x,y) = \sqrt{n+1}\overline{H}_{m,n+1}(x,y)\ ,\ \hat{a}_{0,-}\overline{H}_{m,n}(x,y) = \sqrt{n}\overline{H}_{m,n-1}(x,y)$$

which proves that are the creation and annihilation operators related to the bi-orthogonal Hermite functions $\overline{H}_{m,n}(x,y)$.

By virtue of the above relations we can state the important result concerning the partial differential equation solved by the bi-orthogonal Hermite functions:

$$\left[-\underline{\partial_z}^t \widehat{M}^{-1}\underline{\partial_z} - \left(m+n+1-\frac{1}{4}\underline{z}^t\widehat{M}\underline{z}\right)\right]\overline{H}_{m,n}(x,y) = 0$$

where $\underline{z} = \binom{x}{y}$ and $\underline{\partial_z} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$

Remark: to prove the statement we use the following operational relations:

$$\hat{a}_{+,0}\left[\hat{a}_{-,0}\overline{H}_{m,n}(x,y)\right] = m\overline{H}_{m,n}(x,y)$$

$$\hat{a}_{0,+}\left[\hat{a}_{0,-}\overline{H}_{m,n}(x,y)\right] = n\overline{H}_{m,n}(x,y)$$

which emphasizes the similarity with the orthogonality relation previous established for the ordinary Hermite polynomials:

$$\hat{a}_{+}\hat{a}_{-}he_n(x) = nhe_n(x)$$

To state the analogous result for the functions $\bar{G}_{m,n}(x, y)$ it is sufficient to remind that the adjoint quadratic form:

$$\bar{q}(\underline{z}) = \underline{z}^t \widehat{M}^{-1} \underline{z}$$

which introduced the variable:

$$\underline{\upsilon} = \widehat{M}\underline{z}, \text{ where } \underline{\upsilon} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

allows us to derive the related creation and annihilation operators for the functions $\bar{G}_{m,n}(x, y)$ and then the relevant PDE.

Further investigations

The previous discussed Hermite polynomials and related bi-orthogonal functions satisfy a plethora of relevant relations, for instance in:

Dattoli G. and C.C., *On a new family of Hermite polynomials associated with of parabolic cylinder functions*, 2003, Appl. Math. Comput., 141(1), 143–149

C.C., *Operational methods and new identities for Hermite polynomials*, 2010, Math. Modelling of Natural Phenomena, 12(3), 44-50

C.C., Fornaro C., Vazquez L., *Operational results on bi-orthogonal Hermite functions*, 2016, Acta Math. Universitatis Comenianae, 85 (1), 43-68

but it is much more interesting to explore how the particular bi-orthogonality relation could be extend to other families of classical orthogonal polynomials.

Note that, ordinary and generalized Laguerre and Legendre polynomials, for example, as well as Chebyshev polynomials, could be expressed in terms of polynomials recognized as Hermite. Regarding the generalized Laguerre polynomials:

$$ {}_2L_n(x,y) = H_n(y, D_x^{-1}) $$

where: ${}_2L_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{y^{n-2r} x^r}{(n-2r)!(r!)^2}$, D_x^{-1} the inverse of derivative operator for the Legendre polynomials:

$$ {}_2L_n\left(-\frac{1}{4}(1-y^2), y\right) = P_n(y) $$

where: $P_n(y) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^{n-2r} y^r \left(1-y^2\right)^{n-2r}}{(n-2r)!(r!)^2 2^{2(n-2r)}}$

G. Dattoli, H.M. Srivastava, C.C., *The Laguerre and Legendre Polynomials From an Operational Point of View*, 2001, App. Math and Comp., vol. 124, 117-127 and regarding the Chebyshev polynomials of first and second kind, for $t \in \mathbb{R}$, we have:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n\left(2x, -\frac{1}{t}\right) dt$$

$$T_n(x) = \frac{1}{2(n-1)!}\int_0^{+\infty} e^{-t} t^{n-1} H_n\left(2x, -\frac{1}{t}\right) dt$$

or, more in general:

$$U_n(x, y) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n\left(x, \frac{y}{t}\right) dt$$

and furthermore:

$$U_n^{(m)}(x,y) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n^{(m)}\left(x, \frac{y}{t}\right) dt$$

where:

$$U_n^{(m)}(x,y) = \sum_{r=0}^{[n/m]} \frac{(n-r)! x^{n-mr} y^r}{r!(n-mr)!}, \; H_n^{(m)}(x,y) = \sum_{r=0}^{[n/m]} \frac{n! x^{n-mr} y^r}{r!(n-mr)!}$$

C.C., *Identities and generating functions on Chebyshev polynomials,* 2012, Georgian Math. J., vol. 19, 427-440

C.C., *Multi-dimensional Chebyshev polynomials: a non-conventional approach*, 2019 Comm. in App. and Ind. Math, vol. 10 , 1-19

Thank you

and sorry but *I don't know what a slide rule is for (Sam Cooke)*