ON A CONTINUED FRACTION OF RAMANUJAN

Gaurav Bhatnagar and Mourad E. H. Ismail

OPSFA 2019

Hagenberg, July 21, 2019

12. $$\frac{\Pi(-a^2x^3, x^4)\,\Pi(-b^2x^3, x^4)}{\Pi(-a^2x, x^4)\,\Pi(-b^2x, x^4)} = \frac{1}{1-ab} + \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)} + \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)} + \&c$$

“There is always more in one of Ramanujan’s formulae than meets the eye, as anyone who gets to verify those which look the easiest will soon discover”

–G. H. Hardy

THE VERIFICATION OF ENTRY 12

- 1985: Adiga, Berndt, Bhargava and Watson
  - Acknowledge “help” from Askey and Bressoud
- 1989: Jacobsen (Lorentzen)
- 1987: Ramanathan
  - Proofs use the Bailey-Daum summation, Heine’s transformation, Heine’s continued fraction, contiguous relations

IN THIS TALK: TWO PROOFS

1. Euler’s method
2. The “standard” q-orthogonal polynomial method

EULER'S METHOD (1776)

(36)

DE

TRANSFORMATIONE

SERIEI DIVERGENTIS

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3$$
$$+ m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

IN FRACTIONEM CONTINVAM.

Auctore

L. EVLERO.

Conuent. exhib. d. 11 *Ian.* 1776.

EULER'S METHOD

$$\frac{N}{D} = 1 + \frac{N-D}{D}$$

$$\frac{1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots} = 1 + \frac{(1 + a_1 x + a_2 x^2 + \cdots) - (1 + b_1 x + b_2 x^2 + \cdots)}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}$$

- In Ramanujan 125, we proved all of Ramanujan's q-continued fractions by this method.
- All except for one!

EXAMPLE: WHAT RAMANUJAN DID

Rogers-Ramanujan Continued Fraction

Ramanujan Extended

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

to

$$1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}$$

$$(a;q)_n := \begin{cases} 1 & \text{for } n=0 \\ (1-aq)(1-aq^2)\cdots(1-aq^{n-1}) & \text{for } n=1,2,\ldots. \end{cases}$$

Cor Entry 11.16.15

$$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k}=\cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{1+\cfrac{aq^3}{1+\cdots}}}}$$

$$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k} = \frac{1}{\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}}$$

$$\frac{N}{D} = 1 + \frac{N-D}{D}$$

$$\frac{1}{1+\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k - \sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}}$$

Consider:

$$\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k - \sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k = \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k(1-q^k)$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{q^{k^2}}{(q;q)_k}a^k(1-q^k) &= \sum_{k=1}^{\infty}\frac{q^{k^2}}{(q;q)_{k-1}}a^k \\ &= \sum_{k=0}^{\infty}\frac{q^{(k+1)^2}}{(q;q)_k}a^{k+1} \\ &= aq\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k\end{aligned}$$

We get

$$\frac{1}{1+\dfrac{aq\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}} = \frac{1}{1+\dfrac{aq}{\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}}}$$

Euler's method

$$\frac{N}{D}=1+\frac{N-D}{D}$$

$$\frac{1}{1+\dfrac{aq}{1+\dfrac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k(1-q^k)}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_k}a^k}}}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k}a^k(1-q^k) &= \sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(q;q)_{k-1}}a^k \\ &= \sum_{k=0}^{\infty}\frac{q^{(k+1)^2+k+1}}{(q;q)_k}a^{k+1} \\ &= aq^2\sum_{k=0}^{\infty}\frac{q^{k^2+3k}}{(q;q)_k}a^k\end{aligned}$$

IN GENERAL

$$R(s) := \sum_{k=0}^{\infty} \frac{q^{k^2+sk}}{(q;q)_k} a^k$$

$$\frac{R(s)}{R(s+1)} = 1 + \frac{aq^{s+1}}{\frac{R(s+1)}{R(s+2)}}$$

$$\frac{R(1)}{R(0)} = \frac{1}{\frac{R(0)}{R(1)}} = \frac{1}{1+} \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{s+1}}{\frac{R(s+1)}{R(s+2)}}$$

ENTRY 12

PROOF BY EULER'S METHOD

Define, for $s = 0, 1, 2, 3, \ldots$

$$D(s) := \sum_{k=0}^{\infty} \frac{(bq^{2s-1}/a, -bq/a; q^2)_k}{(q^2, -q^{2s}; q^2)_k} (a^2 q)^k = {}_2\phi_1 \left[\begin{matrix} bq^{2s-1}/a, & -bq/a \\ & -q^2 \end{matrix} ; q^2, \ a^2 q \right]$$

Theorem. For $|q| < 1$ and $|a| < 1$, and $s = 0, 1, 2, 3, \ldots,$ we have

$$\frac{(a^2q^3, b^2q^3; q^4)_\infty}{(a^2q, b^2q; q^4)_\infty} =$$

$$\frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots$$

$$+ \frac{(a-bq^{2s-1})(b-aq^{2s-1})}{(1-ab)(1+q^{2s})} + \frac{(a-bq^{2s+1})(b-aq^{2s+1})}{(1+q^{2s+2})\dfrac{D(s+1)}{D(s+2)}}$$

- Immediately gives "modified convergence"
- Ordinary convergence requires some more work

STEPS

Step I

$$\frac{(a^2q^3, b^2q^3; q^4)_\infty}{(a^2q, b^2q; q^4)_\infty} = \frac{\sum_{k=0}^{\infty} \frac{(bq/a, -bq/a; q^2)_k}{(q^2, -q^2; q^2)_k} (a^2q)^k}{\sum_{k=0}^{\infty} \frac{(b/aq, -b/aq; q^2)_k}{(q^2, -q^2; q^2)_k} (a^2q^3)^k}$$

Using q-binomial theorem

Step II

$$\frac{{}_2\phi_1\left[\begin{matrix} b/aq, & -b/aq \\ & -q^2 \end{matrix}; q^2, \; a^2q^3\right]}{{}_2\phi_1\left[\begin{matrix} bq/a, & -bq/a \\ & -q^2 \end{matrix}; q^2, \; a^2q\right]} = 1 - ab + \frac{(a-bq)(b-aq)}{(1+q^2)\dfrac{D(1)}{D(2)}}$$

Using Euler's Method

Step III

For $s = 0, 1, 2, \ldots,$

$$(1+q^{2s})\frac{D(s)}{D(s+1)} = (1-ab)(1+q^{2s}) + \frac{(a-bq^{2s+1})(b-aq^{2s+1})}{(1+q^{2s+2})\dfrac{D(s+1)}{D(s+2)}}$$

TOUGHEST PART OF THIS PROOF

$$\boxed{\frac{\left(1+bq^{2k+1}/a\right)\left(a^2q^{2s+1}+abq^{2s}\right)}{\left(1+q^{2s}\right)\left(1+q^{2k+2s+2}\right)}=ab+\frac{a\left(aq^{2s+1}-b\right)\left(1-bq^{2k+2s+1}/a\right)}{\left(1+q^{2s}\right)\left(1+q^{2k+2s+2}\right)}}$$

THE “STANDARD” q-ORTHOGONAL POLYNOMIAL METHOD

WE CONSIDER

$$H(x) := \frac{1-ab}{x(1-ab)+(1-ab)+} \frac{(a-bq)(b-aq)}{x(1-ab)+(1-ab)q^2+} \frac{(a-bq^3)(b-aq^3)}{x(1-ab)+(1-ab)q^4+} \frac{(a-bq^5)(b-aq^5)}{x(1-ab)+(1-ab)q^6+\cdots}$$

What we need

$$\boxed{C = \frac{1}{1-ab+} \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+} \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots}$$

$$K = \frac{H(1)}{1-ab} = \frac{1}{2(1-ab)+} \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+} \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots$$

$$\frac{1}{K} - (1-ab) = \frac{1}{C}$$

WE FIND THE VALUE OF H(1)

Theorem. Let $|q| < 1$, $|ab| < 1$ and $|a^2q| < 1$. Then

$$H(1) = \frac{(1-ab)}{2} \cdot \frac{{}_2\phi_1\left[\begin{matrix} -bq/a, \ bq/a \\ -q^2 \end{matrix}; q^2, \ a^2q\right]}{{}_2\phi_1\left[\begin{matrix} -bq/a, \ b/aq \\ -1 \end{matrix}; q^2, \ a^2q\right]}.$$

Corollary. (Ramanujan's Entry II.16.12) Let $|q| < 1$ and $|ab| < 1$. Then, we have

$$\frac{(a^2q^3, b^2q^3; q^4)_\infty}{(a^2q, b^2q; q^4)_\infty} = \frac{1}{1-ab+} \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+} \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots$$

Again uses

$$\boxed{\frac{{}_2\phi_1\left[\begin{matrix} b/aq, \ -b/aq \\ -q^2 \end{matrix}; q^2, \ a^2q^3\right]}{{}_2\phi_1\left[\begin{matrix} bq/a, \ -bq/a \\ -q^2 \end{matrix}; q^2, \ a^2q\right]} = 1 - ab + \frac{(a-bq)(b-aq)}{(1+q^2)\dfrac{D(1)}{D(2)}}}$$

THE STANDARD OP APPROACH

J-fraction $$\frac{A_0}{A_0x+B_0}-\frac{C_1}{A_1x+B_1}-\frac{C_2}{A_2x+B_2}-\cdots$$

Convergent $$\frac{N_k(x)}{D_k(x)}:=\frac{A_0}{A_0x+B_0}-\frac{C_1}{A_1x+B_1}-\cdots-\frac{C_{k-1}}{A_{k-1}x+B_{k-1}}$$

The numerator and denominator polynomials satisfy a 3-term recurrence

$$\begin{aligned}y_{k+1}(x) = ((1-ab)x+(1-ab)q^{2k})y_k(x)+\\ ab(1-bq^{2k-1}/a)(1-aq^{2k-1}/b)y_{k-1}(x),\ \text{for } k>0\end{aligned}$$

$$N_0(x)=0, N_1(x)=1-ab;\ D_0(x)=1, D_1(x)=(1-ab)(x+1)$$

FORMULAS FOR NUMERATOR AND DENOMINATOR

We consider

$$\widehat{D}_k(x) := \frac{D_k(x)}{(bq/a;q^2)_k}$$

Find its generating function

$$\widehat{D}(t) = \frac{(1-b/aq)}{(1-\delta_1 t)(1-\delta_2 t)} \sum_{m=0}^{\infty} \frac{\left(-aqt/b, a^2qt; q^2\right)_m}{(\delta_1 q^2 t, \delta_2 q^2 t; q^2)_m} \left(\frac{b}{aq}\right)^m$$

where,

$$1-(1-ab)xt-abt^2 = (1-\delta_1 t)(1-\delta_2 t).$$

When x = 1 $\delta_1 = 1, \delta_2 = -ab.$

Using Darboux's method

$$\widehat{D}_k(1) \sim \frac{(1-b/aq)}{1+ab}\, {}_2\phi_1\left[\begin{matrix}-aq/b,\ a^2q \\ -abq^2\end{matrix};q^2,\ \frac{b}{aq}\right].$$

Similarly, for the numerator polynomials, and for general x

STEPS IN STANDARD q-OP METHOD

- The numerator and denominator of the convergents are polynomials that satisfy a three-term recurrence relation
- We find the generating function
- We find asymptotic formulas using Darboux's method
- The convergence is due to Markov's theorem

In general

$$xP_k(x) = P_{k+1}(x) + cq^{2k}P_k(x) + \frac{1}{4}(1-bq^{2k-1}/a)(1-aq^{2k-1}/b)P_{k-1}(x), \text{ for } k>0,$$

where

$$c = -\frac{(1-ab)}{2\sqrt{-ab}}.$$

We assume that $P_k(x)$ satisfies the initial conditions

$$P_0(x) = 1, P_1(x) = x - c.$$

$$X(x) = \lim_{k\to\infty} \frac{P_k^*(x)}{P_k(x)}$$

Let $\rho_1 = e^{-i\vartheta}$ $\rho_2 = e^{i\vartheta}$.

Theorem. Let γ_1 and γ_2 given by

$$\gamma_1, \gamma_2 = \frac{aq}{2b}(c \pm \sqrt{c^2-1}),$$

Let F and G be defined as follows:

$$F(\rho) = {}_2\phi_1\left[\begin{matrix} 2\gamma_1\rho,\ 2\gamma_2\rho \\ q^2\rho^2 \end{matrix}; q^2,\ \frac{bq}{a}\right],$$

and

$$G(\rho) = (1-b/aq)\ {}_2\phi_1\left[\begin{matrix} 2\gamma_1\rho,\ 2\gamma_2\rho \\ q^2\rho^2 \end{matrix}; q^2,\ \frac{b}{aq}\right].$$

Then $X(x)$ converges for all complex numbers $x \notin (-1,1)$, except possibly a finite set of points, and is given by

$$X(x) = 2\rho\frac{F(\rho)}{G(\rho)},$$

where ρ is given by:

$$\rho = \begin{cases} \rho_1, & \text{if } \operatorname{Im}(x) > 0, \text{ or } x > 1 \ (x \text{ real}) \\ \rho_2, & \text{if } \operatorname{Im}(x) < 0, \text{ or } x < -1 \ (x \text{ real}) \\ 1, & \text{if } x = 1, \\ -1, & \text{if } x = -1. \end{cases}$$

ON RAMANUJAN

- If Ramanujan had considered this, it would have been easier

$$\boxed{\frac{1}{2(1-ab)} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots.}$$

- But then the answer would not have been so nice
- The first term of Ramanujan's Entry 12 is a bit off
- If Ramanujan began from the product side and used Euler's method, then that would explain why the first term is a bit off
- Indeed, many of Ramanujan's continued fractions are expansions of ratios of series. If there is a product form, it is because the series is summable. (Ramanujan 125 (2014))
- Further, in most of Ramanujan's continued fractions, the first term is a bit off!

"Methods for proving these continued fraction formulas are varied and at times ad hoc. Ramanujan evidently had a systematic procedure for proving these continued fractions, but we don't know what it is."

–Bruce Berndt (2009)

THANK YOU

OPSFA Community, Organizers OPSFA Maryland, Hong Kong, Tianjin, Hagenberg, AMS special sessions,…

In addition: Peter Paule and Christian Krattenthaler