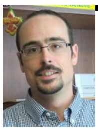
Negative thinking and polynomial analogs



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Abstract: Gaussian binomial coefficients are at the heart of many other polynomial generalizations of interest in special functions, including the q-gamma function and basic hypergeometric series. We discuss the basic properties of Gaussian binomial coefficients and illustrate that many of the combinatorial, analytic and algebraic properties of the usual binomial coefficients permit natural polynomial extensions. On the arithmetic side, various classical congruences extend to the polynomial world. Examples include the famous Lucas congruences

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \pmod{p},$$

where n_i and k_i are the p-adic digits of n and k, as well as Ljunggren's congruences, which state that $\binom{ap}{bp}$ is congruent to $\binom{a}{b}$ modulo p^3 for primes $p \geq 5$.

Through the gamma function, usual binomial coefficients conveniently extend to complex parameters. However, the gamma quotient definition and continuity do not immediately lead to well-defined values for negative integers. In the early 90s, however, Loeb showed that an appropriate natural extension of the binomial coefficients to negative (integer) entries continues to satisfy many of the fundamental properties. In particular, he gave a uniform binomial theorem as well as a combinatorial interpretation in terms of choosing subsets of sets with a negative number of elements. We tell this remarkable, yet little known, story and indicate that all of it can be extended to the case of Gaussian binomial coefficients.

We then discuss various open challenges in the context of supercongruences, series for $1/\pi$, and log-concavity, for which the q-point of view provides a valuable perspective. Time permitting, we will introduce a polynomial analog of the Apéry numbers, the famous sequence which underlies Apéry's proof of the irrationality of $\zeta(3)$. Together with their siblings, known as Apéry-like, they enjoy remarkable properties, including connections with modular forms, and have appeared in various contexts. One of their (still partially conjectural) properties is that these sequences satisfy supercongruences, a term coined by Beukers to indicate that the congruences are modulo exceptionally high powers of primes. We indicate how these congruences, which may be considered as generalizations of Ljunggren's congruences for the binomial coefficients, extend to the polynomial setting as well.

This talk is, in parts, based on joint work with Sam Formichella.