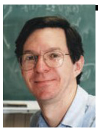
Coefficientwise Hankel-total positivity



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Time: Tuesday 23.07., 09:00, Room AM

Abstract: A matrix M of real numbers is called *totally positive* if every minor of M is nonnegative. Gantmakher and Krein showed in 1937 that a Hankel matrix $H = (a_{i+j})_{i,j\geq 0}$ of real numbers is totally positive if and only if the underlying sequence $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence. Moreover, this holds if and only if the ordinary generating function $\sum_{n=0}^{\infty} a_n t^n$ can be expanded as a Stieltjes-type continued fraction with nonnegative coefficients:

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}$$

(in the sense of formal power series) with all $\alpha_i \geq 0$. So totally positive Hankel matrices are closely connected with the Stieltjes moment problem and with continued fractions.

Here I will introduce a generalization: a matrix M of polynomials (in some set of indeterminates) will be called *coefficientwise totally positive* if every minor of M is a polynomial with nonnegative coefficients. And a sequence $(a_n)_{n\geq 0}$ of polynomials will be called *coefficientwise Hankel-totally positive* if the Hankel matrix $H = (a_{i+j})_{i,j\geq 0}$ associated to (a_n) is coefficientwise totally positive. It turns out that many sequences of polynomials arising naturally in enumerative combinatorics are (empirically) coefficientwise Hankel-totally positive. In some cases this can be proven using continued fractions, by either combinatorial or algebraic methods; I will sketch how this is done. In many other cases it remains an open problem.

One of the more recent advances in this research is perhaps of independent interest to special-functions workers: we have found branched continued fractions for ratios of contiguous hypergeometric series ${}_rF_s$ for arbitrary r and s, which generalize Gauss' continued fraction for ratios of contiguous ${}_2F_1$. For the cases $s = 0$ we can use these to prove coefficientwise Hankel-total positivity.

Reference: Mathias Pétréolle, Alan D. Sokal and Bao-Xuan Zhu, arXiv:1807.03271