MS10: Potential theory and applications to orthogonal polynomials and minimal energy

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Recent applications of Potential Theory to the theory of orthogonal polynomials have allowed for significant advancement of the subject. Methods, such as the Riemann-Hilbert approach, for investigating asymptotic behavior of orthogonal polynomials include prominently the equilibrium measure of a compact set in the complex plane. Another important application of potential theory is to minimal energy problems on the sphere and other manifolds. Seemingly different, both of these areas of analysis explore the convergence properties of discrete potentials. It is the intention of the minisymposium is to provide a common bridge between them and allow for interchanging of ideas.

A new approach for solving equilibrium problems in potential theory

10.01 **Ana Matos**
(*Laboratoire Paul Painlevé, Université de Lille, France*)
Time: Thursday 25.07., 10:30 - 11:00, Room SH 02

Abstract: We are interested in computing the unknown density of an equilibrium problem in logarithmic potential theory where the support of the equilibrium measure is a finite union of distinct intervals. Inspired by a Riemann-Hilbert approach, we reduce the problem to the solution of a system of singular integral equations with Cauchy kernels.

After briefly recalling the well-studied polynomial approach, we will be interested in considering rational approximations of the solutions, expressed in a basis of orthogonal rational functions with prescribed poles. This new approach ensures stable computations. These approximations satisfy some interpolatory conditions. Inspired by the third Zolotareff problem, the poles and the interpolation points are chosen in such a way that we can ensure small errors. We will also discuss a link with recent performant algorithms like the multipole method.

Finally, our new error estimates will be confirmed by numerical results. This is joint work with B. Beckermann.

A family of entire functions connecting the Bessel function J_1 and the Lambert W function

10.02 **Christian Berg**
(*University of Copenhagen, Denmark*)
Time: Thursday 25.07., 11:00 - 11:30, Room SH 02

Abstract: At the 7th OPSFA, Copenhagen 2003, we posed the problem of determining the largest value $\alpha = \alpha^* > 0$ for which $f_\alpha(x) = e^\alpha - (1 + 1/x)^{\alpha x}$, $x > 0$ is a completely monotonic function, and it was noticed that $1 \le \alpha^* < 3$ and that graphs suggest that $\alpha^* > 2$. Numerical estimates given in [2] showed that $\alpha^* \approx 2.29965\,6443$.

We improve this result by combining Fourier analysis with complex analysis to find a family φ_α, $\alpha > 0$, of entire functions such that $f_\alpha(x) = \int_0^\infty e^{-sx}\varphi_\alpha(s)\,ds$ for $x > 0$.

We show that each function φ_α has an expansion in power series, whose coefficients are determined in terms of Bell polynomials. This expansion leads to several properties of the functions φ_α, which turn out to be related to the well known Bessel function J_1 when α is large, and to the Lambert W function when α is small.

On the other hand, by numerically evaluating the series expansion by using the alternating series test, we are able to show the behavior of φ_α as α increases from 0 to ∞ and to obtain a very precise approximation

of α^* such that $\varphi_\alpha(s) \geq 0$, $s > 0$, or equivalently, such that f_α is completely monotonic precisely for $0 < \alpha \leq \alpha^*$. We find $\alpha^* \approx 2.29965\,64432\,53461\,30332$.

The talk is based on the manuscript [1].

[1] C. Berg, E. Massa and A. P. Peron, *A family of entire functions connecting the Bessel function J_1 and the Lambert W function.* ArXiv:1903.07574.

[2] E. Shemyakova, S. I. Khashin and D. J. Jeffrey, *A conjecture concerning a completely monotonic function*, Computers and Mathematics with Applications **60** (2010), 1360–1363.

New bounds for the extreme zeros of classical orthogonal polynomials

10.03 **Geno Nikolov**
(*Sofia University "St. Kliment Ohridski", Bulgaria*)
Time: Thursday 25.07., 11:30 - 12:00, Room SH 02

Abstract: The zeros of classical orthogonal polynomials have been a topic of intensive investigation. There are many reasons for this interest, such as the nice electrostatic interpretation of the zeros of the Jacobi, Laguerre and Hermite polynomials, their important role as nodes of Gaussian quadrature formulae, as well as the key role these zeros play in the proofs of some classical inequalities.

Derivation of sharp upper and lower bounds for the extreme zeros is of particular interest. For this, powerful analytic and discrete techniques have been developed. Among them are Sturms comparison theorem for the zeros of solutions of second order differential equations, A. Markovs theorem on monotonicity of zeros of orthogonal polynomials in terms of the behavior of the weight function, the Hellmann-Feynman theorem on variation of eigenvalues of Hermitian matrices, the Obrechkoff theorem on Descartess rule of signs, the Wall-Wetzel theorem on eigenvalues of Jacobi matrices in terms of chain sequences, etc.

We apply the Euler-Rayleigh method to obtain some new bounds for the extreme zeros of the Jacobi (in particular, of Gegenbauer) and Laguerre polynomials. Typically, the comparison of the different estimates does not single out a "best" bound as these estimates depend on two or three parameters. We show that our bounds improve some of the best known bounds obtained recently.

The logarithmic and Riesz minimal energy problem on sets of revolution — new progress

10.04 **Johann Brauchart**
(*Technische Universität Graz, Austria*)
Time: Thursday 25.07., 12:00 - 12:30, Room SH 02

Abstract: Finding the logarithmic and Riesz equilibrium measure on a set of revolution which is not the sphere proves to be challenging even in the simplest cases of a finite cylinder (revolving line segments) or a circular torus (revolving circle). We discuss theoretical and numerical results. This is joint work with with Doug Hardin and Edward B. Saff.

Minimal energy problem on the sphere, equilibrium support, and quadrature domains

10.05 **Peter Dragnev**
(*Purdue University Fort Wayne, USA*)
Time: Thursday 25.07., 15:30 - 16:00, Room HS 5

Abstract: We consider the minimal logarithmic energy problem on the unit sphere in the presence of external field exerted by finitely many point masses. For relatively weak mass charges the equilibrium support is obtained by removing perfect spherical caps centered at the points. We shall characterize the equilibrium support beyond such interactions and obtain a remarkable connection with quadrature domains in the complex plane. This is joint work with Alan Legg.

Spherical ensemble with two charges

10.06 **Juan G. Criado del Rey**
(*Katholieke Universitait Leuven, Belgium*)
Time: Thursday 25.07., 16:00 - 16:30, Room HS 5

Abstract: We study the weighted equilibrium measure associated to a logarithmic external field generated by two-point charges on the two-dimensional sphere. When the charges are small, the droplet is known to be the complement of two spherical caps. We describe the droplet when the charges are arbitrarily large, focusing on the symmetric case.

Potential theory on orthogonal polynomials arising from subnormal and hyponormal operators

10.07 **Nikos Stylianopoulos**
(*University of Cyprus, Cyprus*)
Time: Thursday 25.07., 16:30 - 17:00, Room HS 5

Abstract: The purpose of the talk is to show how tools from potential theory can be used in obtaining results for the distribution of zeros and the n-th root behavior, of certain orthogonal polynomials associated with subnormal and hyponormal operators.

Hyperuniformity on flat tori

10.08 **Tetiana Stepaniuk**
(*RICAM, Austrian Academy of Sciences, Linz, Austria*)
Time: Thursday 25.07., 17:00 - 17:30, Room HS 5

Abstract: In the talk we study hyperuniformity on flat tori. Hyperuniform point sets on the unit sphere have been studied by J. Brauchart, P. Grabner, W. Kusner and J. Ziee. It is shown that point sets which are hyperuniform for large balls, small balls or balls of threshold order on the flat tori are uniformly distributed. We also show that QMC-designs sequences for Sobolev classes and probabilistic point sets (with respect to jittered samplings) are hyperuniform.