MS08: Asymptotics via non-standard orthogonality

Organizers: Andrei Martínez Finkelshtein (*Baylor University, Texas, USA / Univ. de Almería, Spain*)
Guilherme Silva (*University of Michigan, USA*)
Maxim Yattselev (*Indiana University IUPUI, USA*)

Asymptotic behavior of sequences of polynomials can certainly be called a classical problem. In particular, analytic properties of classical orthogonal polynomials have attracted interest from late 19th century. Nevertheless, in the last few decades the field has experienced several striking developments. On one hand they were stimulated by applications: for instance, many models of mathematical physics can be described in terms of sequences of polynomials exhibiting non-standard type of orthogonality (multiple, non-hermitian, Sobolev, matrix, to mention a few), and their asymptotic analysis is the key to the study of large-scale phenomena. On the other hand, new methods from potential theory, spectral theory and integrable systems have been successfully developed. For this mini-symposium, we plan on bringing together a range of experts in the asymptotic theory of orthogonal polynomials and their generalizations, representing a wide scope of techniques and applications from a modern perspective.

Non-hermitian orthogonality in random tiling problems

08.01 **Arno Kuijlaars**
(*Katholieke Universitait Leuven, Belgium*)
Time: Wednesday 24.07., 10:30 - 11:00, Room HS 3

Abstract: I will discuss how polynomials with a non-hermitian orthogonality on a contour in the complex plane arise in certain random tiling problems. In the case of periodic weightings the setup generalizes to matrix valued orthogonality.

In work with Maurice Duits (KTH Stockholm) the Riemann-Hilbert problem for matrix valued orthogonal polynomials was used to obtain asymptotics for domino tilings of the two-periodic Aztec diamond. This model is remarkable since it gives rise to a gas phase, in addition to the more common solid and liquid phases.

[1] M. Duits and A. B. J. Kuijlaars. The two periodic Aztec diamond and matrix valued orthogonal polynomials. *Preprint arXiv:1712.05636, to appear in J. Eur. Math. Soc.*

Two applications of non-standard orthogonality

08.02 **Alfredo Deaño**
(*University of Kent, UK*)
Time: Wednesday 24.07., 11:00 - 11:30, Room HS 3

Abstract: In this talk we will illustrate the use of non-standard orthogonal polynomials (OPs), more precisely non-Hermitian OPs in the complex plane, and their large degree asymptotics in two different but related contexts: asymptotic behavior of special function solutions of Painlevé differential equations and analysis of non-Hermitian random matrix models with singularities. *Partly based on joint work with Nick Simm (University of Sussex, UK).*

Construction of the global parametrix for the kissing polynomials

08.03 **Andrew Celsus**
(*University of Cambridge, UK*)
Time: Wednesday 24.07., 11:30 - 12:00, Room HS 3

Abstract: When trying to implement the Deift-Zhou method of steepest descent to recover asymptotics of orthogonal polynomials, one needs to construct solutions to a model Riemann-Hilbert problem (RHP).

When studying a certain family of orthogonal polynomials with complex weights known as the kissing polynomials, the model problem does not possess the same symmetries that one usually encounters when dealing with positive weight functions. As such, the construction of the global parametrix, which is the solution of this model problem, requires a different approach. The goal of this talk is to outline the construction of global parametrix which arises when one is trying to study asymptotics of the kissing polynomials. *Joint work with Guilherme Silva of the University of Michigan.*

Generalized Jacobi polynomials on a cross

08.04 **Ahmad Barhoumi**
(*Indiana University IUPUI, USA*)
Time: Wednesday 24.07., 12:00 - 12:30, Room HS 3

Abstract: Polynomials satisfying a non-Hermitian orthogonality relation appear naturally in many places, one of which is the construction of Padé approximants. One feature that sets these polynomials apart from orthogonal polynomials on the real line is that the degree of the polynomial orthogonal up to order n may be less than n. In this talk, I will discuss the asymptotic analysis of a specific family of Jacobi-type polynomials via Riemann–Hilbert Problem while highlighting how the degeneration of degree displays itself in the analysis. *Joint work with Maxim Yattselev.*

Some results and conjectures on the asymptotic properties of polynomials orthogonal over bounded domains

08.05 **Erwin Miña Díaz**
(*University of Mississippi, USA*)
Time: Thursday 25.07., 10:30 - 11:00, Room HS 3

Abstract: In this talk, we will discuss some of the recent progress made in understanding the asymptotic behavior of polynomials orthogonal with respect to the area measure over a bounded domain. We will conjecture what that behavior is in the attractive but elusive case of orthogonality over a domain bounded by a piecewise analytic Jordan curve.

Converting planar orthogonality to orthogonality on a contour

08.06 **Alan Groot**
(*Katholieke Universitait Leuven, Belgium*)
Time: Thursday 25.07., 11:00 - 11:30, Room HS 3

Abstract: We look at a model related to the spherical ensemble, by introducing two points on the sphere that repel particles. The corresponding average characteristic polynomial satisfies a Hermitian orthogonality on the complex plane. We show how to convert this planar orthogonality to non-Hermitian orthogonality with respect to a weight on a collection of contours. *Joint work with Juan Criado del Rey and Arno Kuijlaars.*

Zeros of Faber polynomials for Joukowski airfoils

08.07 **Franck Wielonsky**
(*Centre de Mathématiques et Informatique, Université Aix-Marseille, France*)
Time: Thursday 25.07., 11:30 - 12:00, Room HS 3

Abstract: Let K be the closure of a bounded region in the complex plane with simply connected complement whose boundary is a piecewise analytic curve with at least one outward cusp. The asymptotics of zeros of Faber polynomials for K are not understood in this general setting. Joukowski airfoils provide

a particular class of such sets. We determine the (unique) weak-* limit of the full sequence of normalized counting measures of the Faber polynomials for Joukowski airfoils. This limit is always different from the equilibrium measure of K. This implies that these airfoils admit an electrostatic skeleton and also explains an interesting class of examples of Ullman related to Chebyshev quadrature. *Joint work with Norm Levenberg.*

Detecting outliers with Christoffel-Darboux kernels

08.08 **Bernhard Beckermann**
(*Laboratoire de Mathématiques Paul Painlevé, Université de Lille, France*)
Time: Thursday 25.07., 12:00 - 12:30, Room HS 3

Abstract: Two central objects in constructive approximation, the Christoffel-Darboux kernel and the Christoffel function, are encoding ample information about the associated moment data and ultimately about the possible generating measures. We develop a multivariate theory of the Christoffel-Darboux kernel in $\mathbb{C}^d$, with emphasis on the perturbation of Christoffel functions and their level sets with respect to perturbations of small norm or low rank. The statistical notion of leverage score provides a quantitative criterion for the detection of outliers in large data. Using the refined theory of Bergman orthogonal polynomials, we illustrate the main results, including some numerical simulations, in the case of finite atomic perturbations of area measure of a 2D region. Methods of function theory of a complex variable and (pluri)potential theory are widely used in the derivation of our perturbation formulas. *Joint work with Mihai Putinar (University of California at Santa Barbara), Edward B. Saff (Vanderbilt University) and Nikos Stylianopoulos (University of Cyprus).*

Nikishin systems on star-like sets: limiting functions in ratio asymptotics

08.09 **Abey López García**
(*University of Central Florida, USA*)
Time: Thursday 25.07., 15:30 - 16:00, Room HS 3

Abstract: Let $p \geq 2$ be an integer, and let $\{Q_n\}_{n\geq 0}$ be the sequence of monic, type II multiorthogonal polynomials associated with a Nikishin system of measures supported on a compact subset of the $(p+1)$-star $S_+ = \{z \in \mathbb{C} : z^{p+1} \geq 0\}$. Under some conditions, the sequence of ratios Q_{n+1}/Q_n has periodic limits outside the support of the measures in the Nikishin system. In this talk I will describe the limiting functions in terms of certain conformal algebraic functions defined on a compact Riemann surface of genus zero. *Joint work with G. López Lagomasino.*

On the irrationality and the measure of irrationality of $\log(1 + 1/m) \log(1 - 1/m)$

08.10 **Vladimir Lysov**
(*Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, Russia*)
Time: Thursday 25.07., 16:00 - 16:30, Room HS 3

Abstract: We consider the Diophantine approximants for the product of the two logarithms $\gamma_m := \log\left(1 + \frac{1}{m}\right) \log\left(1 - \frac{1}{m}\right)$ for an integer m. We prove that for all $m \geq 33$ the number γ_m is irrational. This is an improvement of the previous result by M. Hata [1]. We also find new upper estimates of the measure of irrationality of γ_m.

Our approach is based on the Hermite–Padé approximants for the vector of functions (f_1, f_2, f_3), where

$$f_1(z) := \log\left(1 + \frac{1}{z}\right), \quad f_2(z) := \log\left(1 - \frac{1}{z}\right), \quad f_3 := f_1 f_2.$$

This vector is an example of the Generalized Nikishin system of Markov functions on graphs [2]. The common denominator of the approximants satisfies certain multiple orthogonality relations. The key ingredient

of our proof is an explicit formula for the common denominator. By means of this formula we obtain the asymptotics of the sequence of the approximants and also some remarkable arithmetic properties of them.

[1] M. Hata. The irrationality of $\log(1+1/q)\log(1-1/q)$. *Trans. Amer. Math. Soc.* **350**:6 (1998), 2311–2327.

[2] A. I. Aptekarev, V. G. Lysov. Systems of Markov functions generated by graphs and the asymptotics of their Hermite-Padé approximants. *Mat. Sb.* **201**:2 (2010), 183–234.

Properties of some classes of quasi-orthogonal polynomials

08.11

Kerstin Jordaan
(*University of South Africa, Pretoria, South Africa*)
Time: Thursday 25.07., 16:30 - 17:00, Room HS 3

Abstract: In this talk I will prove the quasi-orthogonality of some classes of hypergeometric and q-hypergeometric polynomials that do not appear in the Askey or q-Askey scheme for orthogonal polynomials. The polynomials considered include, as special cases, two ${}_3F_2$ polynomials considered by Dickinson [Proc. Amer. Math. Soc., 12 (1961), 185–194] and a ${}_2F_2$ polynomial with only positive zeros. I will derive three-term recurrence relations and second order differential equations for the quasi-orthogonal polynomials and investigate whether the recurrence coefficients satisfy conditions necessary for orthogonality established recently by Ismail and Wang [J. Math. Anal. Appl., 474(2) (2019), 1178–1197]. The location and interlacing of the real zeros of the polynomials under consideration will also be discussed.

Blaschke Products, numerical ranges, and the zeros of orthogonal polynomials

08.12

Brian Simanek
(*Baylor University, Texas, USA*)
Time: Thursday 25.07., 17:00 - 17:30, Room HS 3

Abstract: Our main object of interest will be the location of zeros of orthogonal polynomials on the unit circle. We will discuss some recent developments that relate these zeros to Blaschke products, numerical ranges of matrices, algebraic curves, quadrature measures, and Poncelet's Theorem. Some open problems will be mentioned along the way.

Asymptotics of orthogonal polynomials with unbounded recurrence coefficients

08.13

Grzegorz Świderski
(*Mathematical Institute, University of Wrocław, Poland*)
Time: Friday 26.07., 10:30 - 11:00, Room HS 3

Abstract: Let μ be a probability measure on the real line with all moments finite. Let $(p_n : n \geq 0)$ be the corresponding sequence of orthonormal polynomials. It satisfies

$$\begin{gathered} p_0(x) = 1, \quad p_1(x) = \frac{x - b_0}{a_0}, \\ a_{n-1}p_{n-1}(x) + b_n p_n(x) + a_n p_{n+1}(x) = x p_n(x) \quad (n \geq 1), \end{gathered} \tag{1}$$

for some sequences $a_n > 0$ and $b_n \in \mathbb{R}$. Conversely, Favard's theorem states that every sequence of polynomials satisfying (**??**) is orthonormal with respect to some measure μ. The measure μ is unique if the Carleman condition is satisfied, i.e. when

$$\sum_{k=0}^{\infty} \frac{1}{a_k} = \infty.$$

In the proposed talk we are interested in the asymptotic behaviour of orthogonal polynomials in terms of its recurrence coefficients. Let $N \geq 1$ be an integer. In the analysis the crucial role is played by the so-called N-step transfer matrix defined by

$$X_n(x) = \prod_{j=n}^{n+N-1} \begin{pmatrix} 0 & 1 \\ -\frac{a_{j-1}}{a_j} & \frac{x-b_j}{a_j} \end{pmatrix}.$$

We are going to present the following

Theorem 1. *Let $N \geq 1$ be a positive integer and let $i \in \{0, 1, \ldots, N-1\}$. Suppose that the sequence $(X_{nN+i} : n \in \mathbb{N})$ is of bounded variation and let $\mathcal{X}_i$ be its limit. Let*

$$\Lambda = \{x \in \mathbb{R} : |\operatorname{tr} \mathcal{X}_i(x)| < 2\}.$$

If $\det \mathcal{X}_i = 1$ and the Carleman condition is satisfied, then the measure μ is purely absolutely continuous on Λ and there is a continuous real-valued function η such that

$$\sqrt{a_{kN+i-1}} p_{kN+i}(x) = \sqrt{\frac{2|[\mathcal{X}_i(x)]_{21}|}{\pi\mu'(x)\sqrt{4-(\operatorname{tr}\mathcal{X}_i(x))^2}}} \sin\Big(\sum_{j=1}^{k} \theta_j(x) + \eta(x)\Big) + \epsilon_k(x), \qquad x \in \Lambda$$

for explicit θ_j and a constructive upper bound on ϵ_k.

Theorem 1 is an extension of results obtained in Máté-Nevai-Totik (1985), Geronimo-Van Assche (1991) and Aptekarev-Geronimo (2016). Our approach is based on uniform diagonalisation of transfer matrices.

We are going to present the applications of Theorem 1 with unbounded a_n to the asymptotics of Christoffel functions with the rate of convergence and to universality limits of Christoffel-Darboux kernel.

This is a joint work with Bartosz Trojan (Polish Academy of Sciences).

Biorthogonal rational functions involving two parameters and the Christoffel type transformation

08.14

Swaminathan Anbhu
(*Indian Institute of Technology, Roorkee, Uttarakhand, India*)

Time: Friday 26.07., 11:00 - 11:30, Room HS 3

Abstract: In this work, a general T-fraction based on a polynomial map is considered. Two generalized linear matrix pencils of the form $\mathcal{G} - z\mathcal{H}$, where $\mathcal{G}$ and $\mathcal{H}$ are tridiagonal matrices, associated to this polynomial map are considered and the orthogonality of the related Laurent polynomials are discussed. These matrix pencils are useful in constructing two sequences of biorthogonal rational functions, $\{p_n^L(z)\}_{n=0}^{\infty}$ and $\{p_n^R(z)\}_{n=0}^{\infty}$, associated with the parameters a_n and b_n respectively, that form the components of the left and right eigenvectors of the matrix pencil. The procedure for constructing these two families is different from the one given in [3]. These two different sequences of orthogonal rational functions lead to the recurrence relations given by

$$\mathcal{P}_{n+1}(z) = \rho_n(z-\nu_n)\mathcal{P}_n(z) + \tau_n(z-a_n)(z-b_n)\mathcal{P}_{n-1}(z), n \geq 1,$$

with initial conditions $\mathcal{P}_0(z) = 1$ and $\mathcal{P}_1(z) = \rho_0(z-\nu_0)$ that are defined on the unit circle as well in the real line. These are known as R_{II} type recurrence relations and were studied by Ismail and Masson [2] and Zhedanov [4] independently. A particular case is considered that provides a Christoffel type transformation of the generalized eigenvalue problem with a reformulation different from the existing literature. Specific illustrations are provided to support the given results.

[1] Kiran Kumar Behera and A. Swaminathan, Biorthogonal rational functions of R_{II} type, Proc. Amer. Math. Soc., (2019), https://doi.org/10.1090/proc/14443, 13 pages.

[2] M. E. H. Ismail and D. R. Masson, Generalized orthogonality and continued fractions, J. Approx. Theory **83** (1995), no. 1, 1–40.
[3] L. Velázquez, Spectral methods for orthogonal rational functions, J. Funct. Anal. **254** (2008), no. 4, 954–986.
[4] A. Zhedanov, Biorthogonal rational functions and the generalized eigenvalue problem, J. Approx. Theory **101** (1999), no. 2, 303–329.

On π_N-coherent pair with index M and order (m,k) of orthogonal polynomial sequences

08.15

Dieudonne Mbouna
(*University of Coimbra, Portugal*)
Time: Friday 26.07., 11:30 - 12:00, Room HS 3

Abstract: Let M and N be non-negative integer numbers, π_N a monic polynomial of degree N, and $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ two monic orthogonal polynomial sequences such that their normalized derivatives of orders m and k (respectively) satisfy

$$\pi_N(x)P_n^{[m]}(x) = \sum_{j=n-M}^{n+N} r_{n,j}Q_j^{[k]}(x)$$

for all $n = 0, 1, 2, \ldots$, where each $r_{n,j}$ is a complex number independent of x. It is shown that under some natural constraints, both $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ belong to the semiclassical orthogonal polynomials class. In addition we show that the corresponding linear functionals with respect to which $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are orthogonal, are also connected by a rational modification (in the distributional sense). This leads to the concept of π_N-coherent pair with index M and order (m,k), as another generalization of the notion of coherent pair of measures introduced by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna [J. Aprox. Theory **65** (1991) 151–175], and subsequently generalized by several authors.

This is a joint work with Renato Alvarez-Nodarse, Kenier Castillo and José Carlos Petronilho.