MS03: Trends on orthogonal polynomials in weighted Sobolev spaces

Organizers: Francisco Marcellán (*Universidad Carlos III de Madrid and ICMAT, Spain*)
Juan José Moreno-Balcázar (*Universidad de Almería, Spain*)

In this MS we will consider different approaches and applications of polynomials orthogonal with respect to inner products in weighted Sobolev spaces. The topics to be covered are interpolation and Fourier projectors and their applications to boundary value problems in one and several variables, asymptotic properties of such polynomials, distribution of zeros, convergence of Sobolev-Fourier expansions, Christoffel functions, moment problems, differential operators with Sobolev orthogonal polynomials as eigenfunctions, among others.

New trends on orthogonal polynomials in Sobolev spaces

03.01

Francisco Marcellán and Juan J. Moreno–Balcázar
(*Universidad Carlos III de Madrid and Universidad de Almería, Spain*)
Time: Monday 22.07., 15:30 - 16:00, Room HS 6

Abstract: In this presentation we will analyze some recent trends in the theory of orthogonal polynomials with respect to Sobolev inner products defined for vector of measures supported either on the real line or the unit circle respectively. Analytical and computational approaches will be discussed. In the first case we will focus the attention on asymptotic properties of such polynomials as well as the distribution of their zeros. In the second one, we will study questions related to boundary value problems for differential equations and Sturm-Liouville theory.

Operational methods in the study of Sobolev-Jacobi polynomials

03.02

Nicolas Behr
(*IRIF, Université Paris Diderot, France*)
Time: Monday 22.07., 16:00 - 16:30, Room HS 6

Abstract: Inspired by ideas from umbral calculus and based on the two types of integrals occurring in the defining equations for the gamma and the reciprocal gamma functions, respectively, we develop a multi-variate version of the so-called umbral image technique. Besides providing a class of new formulae for generalized hypergeometric functions and an implementation of series manipulations for computing lacunary generating functions, our main application of these Â techniques is the study of Sobolev-Jacobi polynomials. Motivated by applications to theoretical chemistry, we moreover present a deep link between generalized normal-ordering techniques introduced by Gurappa and Panigrahi, two-variable Hermite polynomials and our integral-based series transforms. This is joint work with G. Dattoli (ENEA Frascati), G.H.E. Duchamp (Paris 13), Silvia Licciardi (ENEA Frascati) and K.A. Penson (Paris 6).

Bispectral Laguerre and Jacobi type polynomials

03.03

Manuel Domínguez de la Iglesia
(*Universidad Nacional Autónoma de México, Mexico*)
Time: Monday 22.07., 16:30 - 17:00, Room HS 6

Abstract: We study the bispectrality of Laguerre and Jacobi type polynomials, which we define by taking linear combinations of a fixed number of consecutive Laguerre or Jacobi polynomials, respectively. These polynomials are eigenfunctions of higher-order differential operators and include, as particular cases, the Krall-Laguerre and the Krall-Jacobi polynomials. We show that these polynomials always satisfy higher-order recurrence relations (i.e., they are bispectral). We also prove that the Krall-Laguerre and the Krall-Jacobi families are the only Laguerre and Jacobi type polynomials which are orthogonal with respect to a measure in the real line. This is a joint work with Antonio J. Durán.

A new symmetric representation of the differential equation for the Laguerre-Sobolev polynomials

03.04 **Clemens Markett**
(*Aachen University of Technology, Germany*)
Time: Monday 22.07., 17:00 - 17:30, Room HS 6

Abstract: In the enduring fruitful research on orthogonal polynomials in weighted Sobolev spaces, the Laguerre-Sobolev polynomials are playing a prominent role. They are orthogonal with respect to a Sobolev-type inner product associated with the classical Laguerre measure on the positive half-line and two point masses $M, N > 0$ at the origin involving functions and their first derivatives. A particularly useful feature of the Laguerre-Sobolev polynomials is their property to arise as the eigenfunctions of a spectral differential operator which, for any Laguerre parameter $\alpha \in \mathbb{N}_0$, is of finite order $4\alpha + 10$. The main purpose of this talk is to establish a new symmetric form of this differential operator which consists of a number of elementary components depending on α, M, N. In particular, the new representation enables us to deduce the symmetry of the operator in the Sobolev space. This readily recovers the orthogonality of the polynomial eigenfunctions. The present results have strongly been motivated and guided by our recent observations that the higher-order differential equations for the so-called Bochner-Krall orthogonal polynomials possess an elementary symmetric form which differs considerably from the classical Lagrange symmetric form. Among them is the differential equation of order $2\alpha + 2\beta + 6$ for the generalized Jacobi polynomials which will serve us as a pattern to illustrate how the main features of the equation carry over to the Laguerre-Sobolev case. We close with some new promising results on the Jacobi-Sobolev equations.

Sobolev biorthogonal polynomials and the Gauss-Borel factorization

03.05 **Manuel Mañas**
(*Universidad Complutense de Madrid, Spain*)
Time: Tuesday 23.07., 10:30 - 11:00, Room HS 6

Abstract: We explore the Gauss-Borel description of the Sobolev birothogonality. A theory of deformations of Sobolev bilinear forms is proposed. We consider both polynomial deformations and a class of transformations related to the action of linear operators on the entries of a given bilinear form. Christoffel type formulae among new and old polynomial sequences are determined. We also discuss generalized Hankel symmetries.

"Hermitian + nilpotent" = Sobolev moment problem

03.06 **Franciszek Hugon Szafraniec**
(*Jagiellonian University, Kraków, Poland*)
Time: Tuesday 23.07., 11:00 - 11:30, Room HS 6

Abstract: I am going to raise the question assisted by the paper The Sobolev moment problem and Jordan dilations, *J. Math. Anal. Appl.* **444** (2016) 1675-1689 (with Michał Wojtylak). Some discussion toward the Dirichlet moment problem will be provided as well.

Fourier series of Sobolev polynomials for coherent pairs of Jacobi type

03.07 **Judit Mínguez**
(*Universidad de La Rioja, Spain*)
Time: Tuesday 23.07., 11:30 - 12:00, Room HS 6

Abstract: The study of orthogonal polynomials with respect to a Sobolev-type inner product has attracted the interest of many researchers in the last years as we can see in [3]. In [1] and [2], the authors proved convergence and uniform boundedness of the partial sums in some cases for Gegenbauer-Sobolev and Jacobi-Sobolev polynomials. In this work, we are going to study Sobolev orthonormal polynomals for

coherent pairs of measures of Jacobi type, in order to prove uniform boundedness and convergence for the partial sums. This is a joint work with O. Ciaurri.

[1] O. Ciaurri, J. Mínguez Ceniceros, Fourier series of Gegenbauer Sobolev Polynomials *SIGMA* **14** (2018), 024, 11 pages.
[2] O. Ciaurri, J. Mínguez Ceniceros, Fourier series of Jacobi-Sobolev polynomials, *Integral Transforms and Special Functions.* https://doi.org/10.1080/10652469.2018.1560279.
[3] F. Marcellán and Y. Xu, On Sobolev orthogonal polynomials, *Expo. Math.* **33** (2015), 308–352.

Rational approximation and Sobolev orthogonal polynomials

03.08 **Hector Pijeira Cabrera**
(*Universidad Carlos III de Madrid, Spain*)
Time: Tuesday 23.07., 12:00 - 12:30, Room HS 6

Abstract: Let $\{S_n\}_{n=0}^{\infty}$ be the sequence of orthogonal polynomialswith respect to the Sobolev-typeinner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + \sum_{j=1}^{N} \eta_j\, f^{(d_j)}(c_j) g^{(d_j)}(c_j),$$

where μ is in the Nevai class $\mathbf{M}(0,1)$, $\eta_j > 0$, $N, d_j \in \mathbb{Z}_+$ and $\{c_1, \ldots, c_N\} \subset \mathbb{R} \setminus [-1,1]$. Under some restriction of order in the discrete part of $\langle \cdot, \cdot \rangle$, we proof that for n sufficiently large the zeros of S_n are real, simple, $n - N$ of them lie on $(-1,1)$ and each of the mass points c_j "attracts" one of the remaining N zeros. The sequences of associated polynomials $\{S_n^{[k]}\}_{n=0}^{\infty}$ are defined for each $k \in \mathbb{Z}_+$. We prove an analog of the Markov's theorem on rational approximation of some class holomorphic functions and we give an estimate of the "speed" of convergence. This is a joint work with Abel Díaz-González.

Eigenvalues of a differential operator related to classical discrete Sobolev orthonormal polynomials

03.09 **Juan F. Mañas-Mañas**
(*Universidad de Almería, Spain*)
Time: Tuesday 23.07., 15:30 - 16:00, Room HS 6

Abstract: We consider the discrete Sobolev inner product

$$(f, g)_S = \int f(x)g(x)d\mu + M f^{(j)}(c) g^{(j)}(c), \quad j \in \mathbb{N} \cup \{0\}, \quad c \in \mathbb{R}, \quad M > 0,$$

where μ is a classical continuous measure with support on the real line (Jacobi, Laguerre or Hermite). The orthonormal polynomials with respect to this Sobolev inner product are eigenfunctions of a differential operator and obtaining the asymptotic behavior of the corresponding eigenvalues is the principal goal of this talk. This is a joint work with Juan J. Moreno-Balcázar.

On Freud-Sobolev type orthogonal polynomials: asymptotics and zeros

03.10 **Lino Gustavo Garza Gaona**
(*Physics and Mathematics Department, Universidad de Monterrey, Mexico*)
Time: Tuesday 23.07., 16:00 - 16:30, Room HS 6

Abstract: In this contribution we consider sequences of monic polynomials orthogonal with respect to the discrete Sobolev type inner product involving a quartic potential

$$\langle f, g \rangle_1 = \int_{\mathbb{R}} f(x)g(x)|x|^{2\lambda+1} e^{-x^4+tx^2} dx + M_0 f(0)g(0) + M_1 f'(0)g'(0).$$

In particular, we obtain algebraic properties related to their zeros, such as equations of motion with respect to the parameter t, and monotonicity results when M_0, M_1 tend to infinity. We also obtain some asymptotic properties for the coefficients on the recurrence relation that the Sobolev-type orthogonal polynomials satisfy.

On coherence relations between quasi-definite linear functionals and Sobolev orthogonal polynomials

03.11 **Luis Alejandro Molano Molano**
(*Universidad Pedagógica y Tecnológica de Colombia, Duitama, Colombia*)
Time: Tuesday 23.07., 16:30 - 17:00, Room HS 6

Abstract: In this talk we consider the non-coherence relation

$$P_{n+1}^{[i]}(x) + a_n^{[1]} P_n^{[i]}(x) + a_n^{[2]} P_{n-1}^{[i]}(x) + b_n \left(Q_{n+1}(x) + c_n Q_n(x)\right) \\ = \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), \tag{1}$$

where the sequences $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are orthogonal with respect to quasi-definite linear functionals u, v and w, respectively, with $P_k^{[i]}(x) := \frac{P_{k+i}^{(i)}(x)}{(k+1)_i}$, $i = 0, 1$, and $a_n^{[i]} b_n c_n d_n (1 + b_n) \neq 0$, $n \geq 0$. Furthermore the linear functionals u and v are related through the rational relation $\rho u = v$ where $\deg \rho > 1$. We pointed out that (**??**) is linked to the concept of *symmetric* $(1,1)-$*coherent pairs* and, under certain conditions on the linear functionals, it can become a coherence relation. Under such conditions, we analyze an inverse problem associated with (**??**) as well as a direct problem where $i = 1$, $a_n^{[2]}, c_n, d_n = 0$ and $b_n = 1$, for $n \geq 0$. Thereby we exhibit conditions under which the sequence $\{R_n(x)\}_{n\geq 0}$ is orthogonal with respect to a Borel positive measure μ supported on an infinite subset on the real line. The case when $\{Q_n(x)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ are the classical Chebyshev polynomials of the first and second kinds, respectively, is studied, as well as algebraic properties of the monic *Sobolev* polynomials, orthogonal with respect to the *Sobolev inner product*

$$\begin{aligned} \langle p, q \rangle_S &= \int_{-1}^{1} p(x)q(x)(1-x^2)^{-1/2}dx + \lambda_1 \int_{-1}^{1} p'(x)q'(x)(1-x^2)^{1/2}dx \\ &\quad + \lambda_2 \int_{-1}^{1} p''(x)q''(x)d\mu(x), \end{aligned}$$

where $\lambda_1, \lambda_2 > 0$.

Sobolev orthogonal polynomials on the triangle

03.12 **Lidia Fernández**
(*Universidad de Granada, Spain*)
Time: Wednesday 24.07., 10:30 - 11:00, Room HS 6

Abstract: In [1], Yuan Xu generalized the standard bases of orthogonal polynomials on the triangle for negative values of the parameters and he showed Sobolev orthogonality for these polynomials. The purpose of this work is to analyze another family of mutually orthogonal polynomials on the triangle with respect to an inner product which involves some derivatives on one side of it. These polynomials are related with some Christoffel–Darboux transformation of one of the three term recurrence relations that polynomials on the triangle satisfy. Some algebraic and analytic properties will be deduced.

[1] Y. Xu, Approximation and orthogonality in Sobolev spaces on a triangle. *Constr. Approx.* **46** (2017) 349–434.

On Dunkl–Sobolev orthogonal polynomials in the ball involving reflection-invariant weights

03.13 **Leonardo Figueroa**
(*Universidad de Concepción, Chile*)
Time: Wednesday 24.07., 11:00 - 11:30, Room HS 6

Abstract: We investigate properties of spaces of polynomials orthogonal with respect to inner products in the ball involving $\mathbb{Z}_2^d$-invariant weights of the form

$$(1 - \|x\|^2)^{\kappa_{d+1}} \prod_{i=1}^{d} |x_i|^{\kappa_i}$$

and their associated Dunkl differential-difference operators. We deduce orthogonal decompositions of these spaces which then allow for characterizing them as eigenspaces of (weak) Sturm–Liouville-type operators, with approximation-theoretical consequences. This talk is based on joint work with Gonzalo A. Benavides.

Coherent pairs of bivariate orthogonal polynomials

03.14 **Misael Marriaga**
(*Universidad Rey Juan Carlos, Madrid, Spain*)
Time: Wednesday 24.07., 11:30 - 12:00, Room HS 6

Abstract: Coherent pairs of measures were introduced in 1991 and constitute a very useful tool in the study of Sobolev orthogonal polynomials on the real line. In this work, coherence and partial coherence in two variables appear as the natural extension of the univariate case. Given two families of bivariate orthogonal polynomials expressed as polynomial systems, they are a partial coherent pair if there exists a polynomials of the second family can be given as a linear combination of the first partial derivatives of (at most) three consecutive polynomials of the first family. A full coherent pair is a pair of families of bivariate orthogonal polynomials related by means of partial coherent relations in each variable. Consequences of this kind of relations concerning both families of bivariate orthogonal polynomials are studied.