MS02: Hypergeometric functions

Organizer: Diego Dominici (*Johannes Kepler University Linz, Austria*)

In this mini-symposium, we will consider recent developments in the field of hypergeometric functions, including generalized hypergeometric functions, basic hypergeometric functions and elliptic hypergeometric functions. Topics will cover summation formulas, asymptotic expansions, integrals and series, etc.

Large parameter asymptotics for hypergeometric and Legendre functions

02.01 **Adri B. Olde Daalhuis**
(*University of Edinburgh, UK*)
Time: Monday 22.07., 10:30 - 11:00, Room AM

Abstract: Surprisingly, apart from some special cases, simple asymptotic expansions for the associated Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ for large degree ν or large order μ are not available in the literature. In this presentation we will fill this gap by deriving several simple (inverse) factorial expansions for these functions and provide sharp and realistic bounds on their error terms. In the cases that ν is an integer or 2μ is an odd integer, many of these new expansions terminate and provide finite representations in terms of simple functions. Most of these representations appear to be new.

It is well known that the hypergeometric series can be regarded as a large-c asymptotic expansion for the hypergeometric function $F(a, b; c; z)$. We will also present computable bounds for the remainder term of this expansion. To our best knowledge, no such estimates have been given in the literature.

Monodromy of the generalized hypergeometric equation in the Frobenius basis

02.02 **Leslie Molag**
(*Katholieke Universitait Leuven, Belgium*)
Time: Monday 22.07., 11:00 - 11:30, Room AM

Abstract: In his 1961 Ph.D. thesis Levelt gave an explicit construction for determining the monodromy group of the generalized hypergeometric equation. His result yields the explicit form of the monodromy matrices in a specific basis, which turns out to be connected to Mellin-Barnes integrals. In some situations we would like to know the form of these matrices in a different basis though, especially in the arguably most standard basis: the Frobenius basis. To obtain the form of the monodromy matrices in this basis I have followed an approach set out by Beukers. A particular challenge is the maximally unipotent case, where logarithmic terms turn up in the Frobenius basis. The emphasis of this talk will be on elucidating the general explicit form of the monodromy matrices for this case.

New series representation formulas for modified Bessel function of second kind of integer order

02.03 **Dragana Jankov Maširević**
(*Josip Juraj Strossmayer University of Osijek, Croatia*)
Time: Monday 22.07., 11:30 - 12:00, Room AM

Abstract: The main aim of this talk is to present two finite sum representation formulas for the modified Bessel function of the second kind K_n of positive integer order $n \in \mathbb{N}$: one of them including, among K_0, K_1, the generalized hypergeometric function ${}_1F_2$ and another which includes only K_0 and the modified Bessel functions of the first kind I_0 and I_1. Also, the obtained finite sum representations are superior with respect to the known expression in a computational efficiency examination, which will be illustrated on several examples.

Newton diagram for the positivity of ${}_1F_2$ hypergeometric functions and Askey-Szegő problem

02.04 **Yong-Kum Cho**
(*Chung-Ang University, Seoul, South Korea*)
Time: Monday 22.07., 12:00 - 12:30, Room AM

Abstract: Concerning the positivity inequality

$$\text{(P)} \qquad {}_1F_2\left[\begin{array}{c} a \\ b, c \end{array} \middle| -\frac{x^2}{4}\right] \geq 0 \qquad (x > 0),$$

with $a > 0$ fixed, we prove that if the parameter pair (b, c) belongs to certain hyperbolic region in $\mathbb{R}^2_+$ containing the Newton diagram associated to $\{(a + 1/2, 2a), (2a, a + 1/2)\}$, then (P) holds true. As an application, we consider the Askey-Szegő problem, related with

$$\int_0^x t^{-\beta} J_\alpha(t)dt \geq 0 \qquad (x > 0),$$

for which the best possible range of parameters is known in an implicit formulation involving transcendental equations, and obtain the lower and upper bounds for this range of parameters. In addition, we apply our criteria to improve the positivity region for the Lommel functions established by J. Steinig in 1972.

[1] Y.-K. Cho and S.-Y. Chung, *On the positivity and zeros of Lommel functions: Hyperbolic extension and interlacing*, J. Math. Anal. Appl. 470, pp. 898–910 (2019)

[2] Y.-K. Cho, S.-Y. Chung and H. Yun, *An extension of positivity for integrals of Bessel functions and Buhmann's radial basis functions*, Proc. Amer. Math. Soc., Series B, Vol. 5, pp. 25–39 (2018)

[3] Y.-K. Cho, S.-Y. Chung and H. Yun, *Rational extension of the Newton diagram for the positivity of ${}_1F_2$ hypergeometric functions and Askey-Szegő problem*, arXiv:1805.11855, Constr. Approx. (to appear) (2019)

[4] Y.-K. Cho and H. Yun, *Newton diagram of positivity for ${}_1F_2$ generalized hypergeometric functions*, Integral Transforms Spec. Funct. 29, pp. 527–542 (2018)

Symbolic computation for D^n-finite functions

02.05 **Antonio Jiménez Pastor**
(*Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria*)
Time: Monday 22.07., 15:30 - 16:00, Room AM

Abstract: Hypergeometric functions fall naturally into the category of D-finite (or holonomic) functions, being able to obtain a linear differential equation for each ${}_pF_q$ hypergeometric function. We recently extended the concept of D-finite functions (power series satisfying linear differential equations with polynomial coefficients) to the recursive idea of D^n-finite functions (power series satisfying linear differential equation with D^{n-1}-finite coefficients. We will show in this talk the definition of these D^n-finite power series and the main properties they satisfy, providing combinatorial examples and open questions related with hypergeometric functions.

Hypergeometric form of fundamental theorem of calculus

02.06 **Petr Blaschke**
(*Silesian University in Opava, Czech Republic*)
Time: Monday 22.07., 16:00 - 16:30, Room AM

Abstract: We introduce a natural method of computing antiderivatives of a large class of functions which stems from the observation that the series expansion of an antiderivative differs from the series expansion of the corresponding integrand by just two Pochhammer symbols. All antiderivatives are thus, in a sense, “hypergeometric”. And hypergeometric functions are therefore the most natural functions to integrate.

In this talk we would like to make two points: First, the method presented is *easy*. So much so that it can be taught in undergraduate university level. And second: It may be used to prove some of the more challenging examples.

In particular, we show that

$$\int_0^\infty \sqrt[8]{\frac{x^2+8x+8-4(2+x)\sqrt{1+x}}{x^{11}}}\mathrm{d}x = \frac{4\Gamma^2\left(\frac{1}{4}\right)}{3\sqrt{2-\sqrt{2}}\sqrt{\pi}},$$

$$G = \Re\left({}_3F_2\left(\begin{matrix}1 & 1 & 1\\ & 2 & 2\end{matrix};\mathrm{i}\right)\right) = \Im\left([\epsilon^2]\,{}_2F_1\left(\begin{matrix}\epsilon & \epsilon\\ & 1\end{matrix};\mathrm{i}\right)\right) = \frac{1}{8}\left(\psi'\left(\frac{1}{4}\right)-\pi^2\right),$$

where G is the Catalan's constant and

$$\int_0^1 x\ln\frac{1}{1+x^2}K(\mathrm{i}x)\mathrm{d}x = \frac{1}{4\sqrt{2\pi}}\left((2-\ln 2)\Gamma^2\left(\frac{1}{4}\right)+4(\ln 2-4)\Gamma^2\left(\frac{3}{4}\right)\right),$$

where K is the complete elliptic integral of the first kind. All of this using a single technique.

A q-Hurwitz zeta function associated with a q-analogue of Bernoulli polynomials and numbers

02.07 **Zeinab Mansour**
(*Cairo University, Giza, Egypt*)
Time: Monday 22.07., 16:30 - 17:00, Room AM

Abstract: Ismail and Mansour in 2018 introduced a pair of q-analogue of the Bernoulli polynomials through the generating function

$$\frac{e_q(xt)}{e_q(t/2)E_q(t/2)-1} = \sum_{k=0}^\infty b_k(x;q)\frac{t^k}{[k]!}$$
$$\frac{E_q(xt)}{e_q(t/2)E_q(t/2)-1} = \sum_{k=0}^\infty B_k(x;q)\frac{t^k}{[k]!}$$

In this talk we introduce a q-analogue of the Hurwitz-Zeta function and Zeta function and prove that the q-zeta function satisfy the identities

$$\zeta_q(s) = \sum_{k=1}^\infty \xi_k^{s-1}\frac{\operatorname{Cos}\xi_k}{(\operatorname{Sin}_q \xi_k)'}, \qquad \zeta_q(-n) = \frac{B_n(q)}{[n+1]}, \quad n\in\mathbb{N}_0$$

where ξ_k are the positive zeros of the $\operatorname{Sin}_q z = \dfrac{(-iz(1-q);q)_\infty - (iz(1-q);q)_\infty}{2i}$, $B_n(q) = B_n(0;q) = b_n(0;q)$, and $[k] := \frac{1-q^k}{1-q}$. We also extend the results of Lidstone expansions introduced introduced by Ismail and Mansour in [Analysis and Applications, https://doi.org/10.1142/S0219530518500264] for Lidstone expansions.

Functional inequalities for the generalized Wright functions

02.08 **Sourav Das**
(*National Institute of Technology, Jamshedpur, Jharkhand, India*)
Time: Monday 22.07., 17:00 - 17:30, Room AM

Abstract: In this work, a generalization of Wright functions is considered. We derive some man value inequalities for this generalized function, such as Turán-type inequalities, Lazarević-type inequalities, Wilker-type inequalities and Redheffer-type inequalities. Furthermore, we establish monotonicity of ratios for sections of series of these generalized Wright functions, the obtained result is also closely related to Turán-type inequalities. Finally, some other related inequalities are also discussed as a consequence.

Sharp parameter range for interlacing of zeros of same degree Laguerre polynomials

02.09 **Kathy Driver**
(*University of Cape Town, South Africa*)
Time: Tuesday 23.07., 10:30 - 11:00, Room AM

Abstract: The sequence of Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is orthogonal on $(0,\infty)$ with respect to the weight function $e^{-x}x^{\alpha}$ provided $\alpha > -1$. It is known that for each $n \in \mathbb{N}$, the zeros of $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha+t)}(x)$ are interlacing for each t with $0 < t \leq 2$. We show that the t-interval $0 < t \leq 2$ is sharp in order for interlacing to hold for every $n \in \mathbb{N}$.

Hypergeometric transformations based on Hahn and Racah polynomials

02.10 **Robert S. Maier**
(*University of Colorado, Boulder, USA*)
Time: Tuesday 23.07., 11:00 - 11:30, Room AM

Abstract: Much as the Gauss hypergeometric function ${}_2F_1$ satisfies many transformation identities, the function ${}_3F_2$ can be quadratically and cubically transformed. For example, a ${}_3F_2$ with a parametric excess equal to $\frac{1}{2}$ or $-\frac{1}{2}$ may be quadratically transformed to a well-poised ${}_3F_2$ or a very well-poised ${}_4F_3$. Summation identities can be derived from such transformations by the classical technique of equating coefficients, or by Gessel–Stanton pairing. We show that the classical quadratic and cubic transformations of ${}_3F_2$ can extended: in the quadratic case, the parametric excess may be greater than $\frac{1}{2}$ or less than $-\frac{1}{2}$ by any natural number. The transformed functions now become hypergeometric functions of higher order, the added parameters of which make contact with the theory of orthogonal polynomials of a discrete argument. For instance, the added parameters can be the (negated) roots of certain dual Hahn or Racah polynomials, which are defined on a quadratic lattice; or in the cubic case, new polynomials with no evident orthogonal interpretation. Extended versions of summation identities of Whipple and Bailey can be derived from the extended transformations of ${}_3F_2$: for instance, extensions of Dougall's theorem on the sum of a 2-balanced, very well-poised ${}_7F_6(1)$.

A family of hypergeometric orthogonal polynomial sequences that contains all the families in the Askey scheme

02.11 **Luis Verde-Star**
(*Universidad Autónoma Metropolitana, Mexico City, Mexico*)
Time: Tuesday 23.07., 11:30 - 12:00, Room AM

Abstract: We introduce a family $\mathcal{H}$ of hypergeometric orthogonal polynomial sequences determined by three polynomials h, f and g with degrees at most 1, 2, and 3, respectively, where some coefficients of g depend on the coefficients of h and f. The sequences in $\mathcal{H}$ satisfy a generalized difference equation of order one.

We express the orthogonal polynomials using the Newton basis associated with the sequence $f(n)$, for $n \geq 0$, and for every sequence in $\mathcal{H}$ we find an explicit hypergeometric representation, the three-term recurrence relation, and the generalized moments with respect to the Newton basis. When f is constant we obtain the classical orthogonal sequences. When f is not constant the sequences satisfy a discrete orthogonality relation and we find the discrete weight function.

The recurrence coefficients for each one of the 15 families of polynomial sequences in the Askey scheme of hypergeometric orthogonal polynomials are obtained by direct substitution of particular values for the parameters in our general formulas.

A Bailey type factorization of Horn's H4 hypergeometric function

02.12 **Carlo Verschoor**
(*University of Utrecht, Netherlands*)
Time: Tuesday 23.07., 12:00 - 12:30, Room AM

Abstract: A well known identity by Bailey states that Appell's F4 function can be written as the product of two Gauss hypergeometric functions under a suitable specialization of the parameters. Other identities of this type are known for Appell's F2 and F3, which are closely related to Bailey's identity. The aim of this talk is to show that the same can be done for Horn's H4 function.