# MS01: Orthogonal polynomials, special functions, and functional equations

Organizers: Walter van Assche (*Katholieke Universiteit Leuven, Belgium*)
Galina Filipuk (*University of Warsaw, Poland*)
Yoshishige Haraoka (*Kumamoto University, Japan*)

In this mini-symposium we would like to gather together experts on linear special functions (e.g., hypergeometric) and nonlinear special functions (e.g., Painlevé equations) to discuss different aspects of these functions and recent advances in differential, difference and $q$-difference equations. One aspect is the relation between orthogonal polynomials and the Painlevé equations and the interest will be in discrete Painlevé equations and special solutions of the Painlevé differential equations that appear in the analysis of orthogonal polynomials.

## Invariants of difference equations and transformation formulae for hypergeometric functions

**01.01** **Akihito Ebisu**
(*Faculty of Information and Computer Science, Chiba Institute of Technology, Japan*)
**Time:** Tuesday 23.07., 15:30 - 16:00, Room AM

**Abstract:** We introduce invariants of linear difference equations. Moreover, using these invariants, we develop a systematic method for constructing transformation formulae for hypergeometric functions. By applying this method to hypergeometric functions, not only known formulae, such as algebraic transformation formulae and transformation formulae of special values, but also new-type transformation formulae are obtained.

## Discrete Laplace method and hypergeometric continued fractions

**01.02** **Katsunori Iwasaki**
(*Hokkaido University, Japan*)
**Time:** Tuesday 23.07., 16:00 - 16:30, Room AM

**Abstract:** A discrete analogue of Laplace's method for hypergeometric series containing a large parameter is developed. It aims to calculate large parameter asymptotics of the series by regarding the sum as a "discrete integral". It is particularly useful when Euler's integral representation of the series diverges so that the classical (continuous) Laplace method for integrals is not available. As an application leading asymptotics of the truncation error for Gauss's ${}_2F_1$ continued fraction is determined exactly, as well as for an infinite number of ${}_3F_2(1)$ continued fractions. Some discussions are also made about contiguous relations from which hypergeometric continued fractions are derived. This is in part a joint work with Akihito Ebisu. A more detailed account can be found in arXiv:1904.03350v1, Ramanujan J. 49 (2019), no.1, 159-213, and J. Math. Anal. Appl. 463 (2018), no.2, 593-610.

## Some special functions of matrix integral type and quantum Painlevé equations

**01.03** **Hironobu Kimura**
(*Graduate school of Science and Technology, Kumamoto University, Japan*)
**Time:** Tuesday 23.07., 16:30 - 17:00, Room AM

**Abstract:** We consider several special functions defined by matrix integrals on Hermitian matrix space, namely, analogues of Gauss hypergeomeric, Kummer's confluent hypergeometric, Bessel, Hermite-Weber and Airy function. We would like to discuss the relation of these functions of matrix integral type with some semi-classical orthogonal polynomials and with the polynomial solutions of quantum Painlevé equations.

### Asymptotic analysis for a confluent KZ type equation

**01.04** **Yoshishige Haraoka**
(*Department of Mathematics, Kumamoto University, Japan*)
**Time:** Tuesday 23.07., 17:00 - 17:30, Room AM

**Abstract:** We are interested in the asymptotic analysis of integrable connections of irregular singular type in several variables. Majima (LNM 1075, Springer, 1984) gave a fundamental idea of asymptotic expansion in several variables, and developed a general theory. However, there seems no essential example where Majima's asymptotic expansion is calculated. The asymptotic analysis in several variables seems to be difficult because there were few examples of integrable connections. In applying the Katz theory on rigid local systems, we get a way of constructing integrable connections in a recursive way, and can obtain infinitely many examples. For example, we obtain a confluent KZ type equation from an integrable connection satisfied by Appell's hypergeometric series $F_4$. We show how we can get Majima's asymptotic expansions for the connection.

### More precise symmetric descriptions for properties of the Askey-Wilson Polynomials and their symmetric sub-families

**01.05** **Howard Cohl**
(*National Institute of Standards and Technology, Gaithersburg, Maryland, USA*)
**Time:** Wednesday 24.07., 10:30 - 11:00, Room AM

**Abstract:** We give a more precise symmetric parametric description for various properties of the Askey-Wilson polynomials including hypergeometric and $q$-integral representations. This description is then applied to connection relations and to generating functions. We produce new $q$-integrals and basic hypergeometric transformations when applied to known generating functions for the symmetric families. This method is applied to produce generating functions and generalized generating functions in the two regimes of $q$ and $1/q$ for $0 < |q| < 1$. This is joint work with Roberto S. Costas-Santos.

### On algebraic independence of solutions for systems of algebraic Mahler functional equations

**01.06** **Hiroshi Ogawara**
(*Kumamoto University, Japan*)
**Time:** Wednesday 24.07., 11:00 - 11:30, Room AM

**Abstract:** Many studies of functional equations of Mahler type have been conducted for algebraic independence of their solutions and algebraic independence of their special values. In this talk, we give criteria for algebraic independence of solutions for a new class of systems of algebraic Mahler functional equations. As an application, we show algebraic independence of the solutions of infinite product forms. We will speak of algebraic independence of special values of our solutions and asymptotic analysis of Mahler functions by using perturbation methods.

### Connection problem of GKZ hypergeometric functions

**01.07** **Saiei-Jaeyeong Matsubara-Heo**
(*Kobe University, Japan*)
**Time:** Wednesday 24.07., 11:30 - 12:00, Room AM

**Abstract:** GKZ (Gelfand, Kapranov, Zelevinsky) system is a holonomic system which describes classical hypergeometric systems in a unified manner. The study of GKZ system is, as a principle, controlled by the combinatorics of the Newton polytope. A typical manifestation of such a mechanism is the description of a connection problem of GKZ hypergeometric functions between "nearby toric infinities". It is expected

that the resulting connection formulae give a method of computing the monodromy group. In this talk, we formulate the connection problem for series solutions when GKZ system is regular holonomic. We give an explicit connection formula described by combinatorics of regular triangulations of the Newton polytope.

## Discrete Painlevé Equations in Tiling Problems

**01.08** **Anton Dzhamay**
(*School of Mathematical Sciences, University of Northern Colorado, Greeley, USA*)
**Time:** Wednesday 24.07., 12:00 - 12:30, Room AM

**Abstract:** The notion of a *gap probability* is one of the main characteristics of a probabilistic model. In [4] A. Borodin, extending to the discrete case a well-known relationswhip between gap probabilities and differential Painlevé equations, showed that for some discrete probabilistic models of Random Matrix Type discrete gap probabilities can be expressed through solutions of discrete Painlevé equations. Many examples of such correspondence for the gap probabilities of discrete polynomial ensembles are listed in [2] and are based on the formalism of discrete Riemann-Hilbert problem and its connection with the isomonodromic deformations of $d$-connections on vector bundles, [1].

In this work we generalize the results of A. Knizel [5] and consider a probabilistic model of random surfaces known as boxed plane partitions (or, equivalently, of lozenge tilings of a hexagon). When equipped with the uniform distribution, this model can be connected with the *Hahn* discrete orthogonal polynomial ensemble. In [3] A. Borodin, V. Gorin, and E. Rains proposed a very general multi-parameter probability weights for such partitions that, in certain limits, map to the $q$-Racah discrete polynomial ensemble, and from that to Racah, $q$-Hahn, and Hahn ensembles. This degeneration matches the degeneration in the classification scheme for discrete Painlevé equation that is due to H. Sakai, [7], as shown below. One of our goals is to understand this matching, including the degenerations.

We begin by describing the moduli space of $q$-connections that correspond to the $q$-Racah probability distribution and match it with the Space of Initial Conditions of discrete Painlevé equations of surface type $A_1^{(1)}$ and symmetry type $E_7^{(1)}$. Then, using the geometric techniques of Sakai's theory, such as the Period Map and the Birational Representation of Affine Weyl Groups, we show how to find a highly non-trivial change of variables from the original spectral (isomonodromic) coordinates to the Painlevé coordinates in which our dynamics matches the standard dynamics as written in [6]. This enables the effective computation of discrete gap probabilities for this model. Further, this change of variables is compatible with the parameter degeneration in both the weight degeneration cascade and the discrete Painlevé degeneration cascade. Finally, the isomonodromic problem gives is a new symmetric Lax pair for the $q$-P$(A_1^{(1)})$ equation.

[1] D. Arinkin and A. Borodin, *Moduli spaces of d-connections and difference Painlevé equations*, Duke Math. J. **134** (2006), no. 3, 515–556.

[2] Alexei Borodin and Dmitriy Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, Comm. Math. Phys. **234** (2003), no. 2, 287–338.

[3] Alexei Borodin, Vadim Gorin, and Eric M. Rains, *q-distributions on boxed plane partitions*, Selecta Math. (N.S.) **16** (2010), no. 4, 731–789.

[4] Alexei Borodin, *Discrete gap probabilities and discrete Painlevé equations*, Duke Math. J. **117** (2003), no. 3, 489–542.

[5] Alisa Knizel, *Moduli spaces of q-connections and gap probabilities*, International Mathematics Research Notices (2016), no. 22, 1073–7928.

[6] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada, *Geometric aspects of Painlevé equations*, J. Phys. A **50** (2017), no. 7, 073001, 164.

[7] Hidetaka Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.

## Orthogonal polynomials with ultra-exponential weight functions: an explicit solution to the Ditkin-Prudnikov problem

**01.09** **Semyon Yakubovich**
(*Faculdade da Ciências, Universidade do Porto, Portugal*)
**Time:** Thursday 25.07., 10:30 - 11:00, Room AM

**Abstract:** In this talk we give an interpretation of new sequences of orthogonal polynomials with ultra-exponential weight functions in terms of the so-called composition orthogonality. The 3-term recurrence relations, explicit representations, generating functions and Rodrigues-type formulae are derived. The method is based on differential properties of the involved special functions and their representations in terms of the Mellin-Barnes and Laplace integrals. Certain advantages of the composition orthogonality are shown to find a relationship with the corresponding multiple orthogonal polynomial ensembles.

## Asymptotics of solutions to the second Painlevé hierarchy

**01.10** **Weiying Hu**
(*Department of Mathematics, City University of Hong Kong*)
**Time:** Thursday 25.07., 11:00 - 11:30, Room AM

**Abstract:** In this talk I will introduce the asymptotics of Hastings-McLeod solutions and Ablowitz-Segur solutions as $x \to \pm\infty$ for any $\alpha$, the parameter appears in the equations. Moreover, the connection formulas for the Ablowitz-Segur solutions as $x \to \pm\infty$ are also derived. The method I use is the Deift and Zhou nonlinear steepest descent approach based on Riemann-Hilbert problem.

## On some families of exactly solvable Schrödinger operators

**01.11** **Jan Derezinski**
(*University of Warsaw, Faculty of Physics, Poland*)
**Time:** Thursday 25.07., 11:30 - 12:00, Room AM

**Abstract:** I will discuss various realizations of 1-dimensional Schrödinger operators with $1/x^2$ and $1/x$ potentials as closed operators on $L^2[0,\infty[$. Their resolvents can be expressed in terms of various kinds of Bessel and Whittaker functions. It is natural to organize them into holomorphic families, allowing for complex coupling constants. Their properties are sometimes quite surprising.

## Exceptional extensions of some $q = -1$ classical orthogonal polynomials

**01.12** **Yu Luo**
(*Graduate School of Informatics, Kyoto University, Japan*)
**Time:** Thursday 25.07., 12:00 - 12:30, Room AM

**Abstract:** In recent years, significant progress on exceptional orthogonal polynomial systems has been made by researchers from mathematical and physical aspects. It was proved that every system of exceptional orthogonal polynomials can be obtained from a classical orthogonal polynomial system through a sequence of Darboux transformations. Note that the classical orthogonal polynomials mentioned in this context are the Hermite, Laguerre and Jacobi polynomials, sometimes also be referred to as the "very" classical orthogonal polynomials. In general, polynomials in the Askey-Wilson scheme all can be called classical. The term classical means that apart from a three-term recurrence relation, these polynomials satisfy also an eigenvalue equation. Recently, several new families of polynomial systems which appear by taking a nontrivial limit $q = -1$ on orthogonal polynomials from the Askey-Wilson scheme have been identified classical. They satisfy eigenvalue problems with differential/difference operators of Dunkl type. Specifically, these polynomials are the Bannai-Ito polynomials, the big $-1$-Jacobi and the little $-1$-Jacobi polynomials. Unlike the previous cases, the associated Dunkl-type operators are of first-order which cannot

be factorized into two first-order as it was performed in an ordinary Darboux transformation. Therefore, we apply a generalized Darboux transformation by making use of a pair of intertwining relations satisfied by the Dunkl-type operators. In this way we derive the exceptional extensions of these $q = -1$ polynomials. An interesting fact of these exceptional orthogonal polynomial systems is that in several cases the corresponding degree sequences are consist of even numbers only, for example, $\{0, 2, 2, 4, 4, \dots\}$. We further study their ladder operators and the associated algebraic relations to address this fact. This is joint work with Satoshi Tsujimoto, Luc Vinet, and Alexei Zhedanov.

### The Matrix Bochner Problem

**01.13** **William Riley Casper**
(*Louisiana State University, Baton Rouge, Louisiana, USA*)
**Time:** Thursday 25.07., 15:30 - 16:00, Room AM

**Abstract:** We present a solution of the matrix Bochner problem, a long-standing open problem in the theory of orthogonal polynomials, with applications to diverse areas of research including representation theory, random matrices, spectral theory, and integrable systems. Our solution is based on ideas applied by Krichever, Mumford, Wilson and others, wherein the algebraic structure of an algebra of differential operators influences the values of the operators in the algebra. By using a similar idea, we convert the matrix Bochner problem to one about noncommutative algebras of GK dimension 1 which are module finite over their centers. Then the problem is resolved using the representation theory of these algebras.

### Wronskian Appell polynomials and symmetric functions

**01.14** **Niels Bonneux**
(*Katholieke Universitait Leuven, Belgium*)
**Time:** Thursday 25.07., 16:00 - 16:30, Room AM

**Abstract:** We study Wronskians of Appell polynomials indexed by integer partitions. These families of polynomials appear in rational solutions of certain Painlevé equations and in the study of exceptional orthogonal polynomials. We determine their derivatives, their average and variance with respect to Plancherel measure, and introduce several recurrence relations. Our proofs all exploit strong connections with the theory of symmetric functions.

### Coefficients of Wronskian Hermite polynomials

**01.15** **Marco Stevens**
(*Katholieke Universitait Leuven, Belgium*)
**Time:** Thursday 25.07., 16:30 - 17:00, Room AM

**Abstract:** Wronskians of Hermite polynomials appear in the rational solutions of Painlevé equations and in the field of exceptional orthogonal polynomials. In previous work, we exploited the combinatorial framework of these polynomials to derive recurrence relations in terms of integer partitions. In this talk, we use this framework to relate the coefficients of Wronskian Hermite polynomials to 2-cores and 2-quotients of integer partitions, as well as to characters of irreducible representations of the symmetric group. Joint work with Niels Bonneux (KU Leuven) and Clare Dunning (University of Kent).

### Exceptional orthogonal polynomials from the semi-classical point of view

**01.16** **Roberto S. Costas Santos**
(*Universidad de Alcalá, Madrid, Spain*)
**Time:** Thursday 25.07., 17:00 - 17:30, Room AM

**Abstract:** In this talk we briefly discuss these the different types of Laguerre and Jacobi exceptional polynomials and obtain some algebraic properties for these families by using certain first order differential operators. We connect these families with certain semiclassical orthogonal polynomials. This is joint work with Jessica Stewart Kelly.

### Nielsen's beta-function and some infinitely divisible distributions

**01.17** **Henrik Laurberg Pedersen**
(*Department of Mathematical Sciences, University of Copenhagen, Denmark*)
**Time:** Friday 26.07., 10:30 - 11:00, Room AM

**Abstract:** Nielsen's beta-function is a classical special function related to Euler's gamma function. It is by definition a completely monotonic function. We obtain that it is a so-called logarithmically completely monotonic function, and determine the corresponding family of infinitely divisible distributions. This is based on the Steutel-Kristiansen theorem, relating generalized Stieltjes functions of positive order with logarithmically completely monotonic functions.

These results are related to Laplace transforms of positive, even and periodic functions. A method supplying us with a number of concrete examples of logarithmically completely monotonic functions is described.

The talk is based on joint work with Christian Berg (University of Copenhagen) and Stamatis Koumandos (University of Cyprus).

### On the Heun functions

**01.18** **Galina Filipuk**
(*University of Warsaw, Poland*)
**Time:** Friday 26.07., 10:30 - 11:00, Room HS 5

**Abstract:** In this talk I shall present some properties of the Heun functions.

### Recurrence equations and their classical orthogonal polynomial solutions on a quadratic or a $q$-quadratic lattice

**01.19** **Daniel Duviol Tcheutia**
(*Institute of Mathematics, University of Kassel, Germany*)
**Time:** Friday 26.07., 11:00 - 11:30, Room AM

**Abstract:** If $(p_n(x))_{n\geq 0}$ is an orthogonal polynomial system, then $p_n(x)$ satisfies a three-term recurrence relation of type

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \quad (n = 0, 1, 2, \ldots, p_{-1} \equiv 0),$$

with $C_n A_n A_{n-1} > 0$. On the other hand, Favard's theorem states that the converse is true. A general method to derive the coefficients $A_n$, $B_n$, $C_n$ in terms of the polynomial coefficients of the divided-difference equations satisfied by orthogonal polynomials on a quadratic or $q$-quadratic lattice is recalled. If a three-term recurrence relation is given as input, the Maple implementations `rec2ortho` of Koorwinder and Swarttouw (1996) or `retode` of Koepf and Schmersau (2002) can identify its solution which is a (linear transformation of a) classical orthogonal polynomial system of a continuous, a discrete or a $q$-discrete variable, if applicable. The two implementations `rec2ortho` and `retode` do not handle classical orthogonal polynomials on a quadratic or $q$-quadratic lattice. Motivated by an open problem, submitted by Alhaidari during the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications, which will serve as application, the Maple implementation `retode` of Koepf and Schmersau is extended to cover classical orthogonal polynomial solutions on quadratic or $q$-quadratic lattices of three-term recurrence relations.

## On connection problem of $q$-conformal blocks and its application

**01.20** **Hajime Nagoya**
(*Kanazawa University, Ishikawa, Japan*)
**Time:** Friday 26.07., 11:30 - 12:00, Room AM

**Abstract:** Our $q$-conformal block function is derived from an expectation value of intertwiners between Fock modules of Ding-Iohara-Miki algebra. It is a series with explicit coefficients and can be identified with the $q$-Nekrasov partition function. I will explain how we solve the connection problem of the $q$-conformal block with the degenerate condition. Using the connection formula, I construct a fundamental solution of a $q$-linear difference system and obtain tau functions of a $q$-Painlevé system whose limit is the Fuji-Suzuki-Tsuda system.