Discrete Painlevé Equations in Tiling Problems

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Abstract: The notion of a *gap probability* is one of the main characteristics of a probabilistic model. In [4] A. Borodin, extending to the discrete case a well-known relationswhip between gap probabilities and differential Painlevé equations, showed that for some discrete probabilistic models of Random Matrix Type discrete gap probabilities can be expressed through solutions of discrete Painlevé equations. Many examples of such correspondence for the gap probabilities of discrete polynomial ensembles are listed in [2] and are based on the formalism of discrete Riemann-Hilbert problem and its connection with the isomonodromic deformations of d-connections on vector bundles, [1].

In this work we generalize the results of A. Knizel [5] and consider a probabilistic model of random surfaces known as boxed plane partitions (or, equivalently, of lozenge tilings of a hexagon). When equipped with the uniform distribution, this model can be connected with the *Hahn* discrete orthogonal polynomial ensemble. In [3] A. Borodin, V. Gorin, and E. Rains proposed a very general multi-parameter probability weights for such partitions that, in certain limits, map to the q-Racah discrete polynomial ensemble, and from that to Racah, q-Hahn, and Hahn ensembles. This degeneration matches the degeneration in the classification scheme for discrete Painlevé equation that is due to H. Sakai, [7], as shown below. One of our goals is to understand this matching, including the degenerations.

We begin by describing the moduli space of q-connections that correspond to the q-Racah probability distribution and match it with the Space of Initial Conditions of discrete Painlevé equations of surface type $A_1^{(1)}$ and symmetry type $E_7^{(1)}$. Then, using the geometric techniques of Sakai's theory, such as the Period Map and the Birational Representation of Affine Weyl Groups, we show how to find a highly non-trivial change of variables from the original spectral (isomonodromic) coordinates to the Painlevé coordinates in which our dynamics matches the standard dynamics as written in [6]. This enables the effective computation of discrete gap probabilities for this model. Further, this change of variables is compatible with the parameter degeneration in both the weight degeneration cascade and the discrete Painlevé degeneration cascade. Finally, the isomonodromic problem gives is a new symmetric Lax pair for the q-P$(A_1^{(1)})$ equation.

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[2] Alexei Borodin and Dmitriy Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, Comm. Math. Phys. **234** (2003), no. 2, 287–338.

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[4] Alexei Borodin, *Discrete gap probabilities and discrete Painlevé equations*, Duke Math. J. **117** (2003), no. 3, 489–542.

[5] Alisa Knizel, *Moduli spaces of q-connections and gap probabilities*, International Mathematics Research Notices (2016), no. 22, 1073–7928.

[6] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada, *Geometric aspects of Painlevé equations*, J. Phys. A **50** (2017), no. 7, 073001, 164.

[7] Hidetaka Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.