Rational solutions of Painlevé equations

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Abstract: The six Painlevé equations, whose solutions are called the Painlevé transcendents, were derived by Painlevé and his colleagues in the late 19th and early 20th centuries in a classification of second order ordinary differential equations whose solutions have no movable critical points. In the 18th and 19th centuries, the classical special functions such as Bessel, Airy, Legendre and hypergeometric functions, were recognized and developed in response to the problems of the day in electromagnetism, acoustics, hydrodynamics, elasticity and many other areas. Around the middle of the 20th century, as science and engineering continued to expand in new directions, a new class of functions, the Painlevé functions, started to appear in applications. The list of problems now known to be described by the Painlevé equations is large, varied and expanding rapidly. The list includes, at one end, the scattering of neutrons off heavy nuclei, and at the other, the distribution of the zeros of the Riemann-zeta function on the critical line $\mathrm{Re}(z) = \frac{1}{2}$. Amongst many others, there is random matrix theory, the asymptotic theory of orthogonal polynomials, self-similar solutions of integrable equations, combinatorial problems such as the longest increasing subsequence problem, tiling problems, multivariate statistics in the important asymptotic regime where the number of variables and the number of samples are comparable and large, and also random growth problems.

The Painlevé equations possess a plethora of interesting properties including a Hamiltonian structure and associated isomonodromy problems, which express the Painlevé equations as the compatibility condition of two linear systems. Solutions of the Painlevé equations have some interesting asymptotics which are useful in applications. They possess hierarchies of rational solutions and one-parameter families of solutions expressible in terms of the classical special functions, for special values of the parameters. Further the Painlevé equations admit symmetries under affine Weyl groups which are related to the associated Bäcklund transformations.

In this talk I shall discuss rational solutions of Painlevé equations. Although the general solutions of the six Painlevé equations are transcendental, all except the first Painlevé equation possess rational solutions for certain values of the parameters. These solutions are usually expressed in terms of logarithmic derivatives of special polynomials that are Wronskians, often of classical orthogonal polynomials such as Hermite and Laguerre. It is also known that the roots of these special polynomials are highly symmetric in the complex plane. The polynomials arise in applications such as random matrix theory, vortex dynamics, in supersymmetric quantum mechanics, as coefficients of recurrence relations for semi-classical orthogonal polynomials and are examples of exceptional orthogonal polynomials.

In particular, I shall discuss rational solutions of the $A_2^{(1)}$-Painlevé system

$$\begin{aligned} f_0' + f_0(f_1 - f_2) &= \alpha_0, \\ f_1' + f_1(f_2 - f_0) &= \alpha_1, \\ f_2' + f_2(f_0 - f_1) &= \alpha_2, \end{aligned}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$, with α_0, α_1 and α_2 constants, which is equivalent to the fourth Painlevé equation, and describe some new rational solutions of the $A_4^{(1)}$-Painlevé system

$$\begin{aligned} f_0' + f_0(f_1 - f_2 + f_3 - f_4) &= \alpha_0, \\ f_1' + f_1(f_2 - f_3 + f_4 - f_0) &= \alpha_1, \\ f_2' + f_2(f_3 - f_4 + f_0 - f_1) &= \alpha_2, \\ f_3' + f_3(f_4 - f_0 + f_1 - f_2) &= \alpha_3, \\ f_4' + f_4(f_0 - f_1 + f_2 - f_3) &= \alpha_4, \end{aligned}$$

with $\alpha_0, \alpha_1, \ldots, \alpha_4$ constants, which is the second member of the hierarchy.