An Algorithm to Factorize in a Quotient Ring

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Abstract

The motivation for my study is that some examples of graded factorial rings different from the polynomial ring are known. Classical examples of varieties whose coordinate ring is a graded factorial domain include generic surfaces of $\mathbb{P}^3_K$ with order $m \geq 4$, non singular quadrics of $\mathbb{P}^n_K$ $(n \geq 4)$, and Grassmannians are (see (No), (Na), and (Sa)). Moreover, it possible to construct more examples graded factorial domains taking $A(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T]$ where X is an integral, normal, projective scheme defined over a field K whose divisor class group is $\mathrm{Cl}(X) = \mathbb{Z}$, and D is a well defined Weil divisor with rational coefficients (see (Ro), (Da)). Since all these examples are finitely generated K-algebras, it is natural to ask for a method to compute the factorization of an element in a factorial quotient ring R/I where $R = K[X_1, \ldots, X_n]$ is a polynomial ring over a field K, and I is a homogeneous ideal of R to respect to a positive graduation of R.

First I show how to compute the greatest common divisor of two elements in such a graded quotient ring by computing a particular module of syzygies of elements in the polynomial ring $K[X_1, \ldots, X_n]$.

Then I prove that the computation of the factorization of the residue class of F in $K[X_1, \ldots, X_n]/I$ can be reduced to the computation of the minimal primes of the ideal $(F) + I$ in $K[X_1, \ldots, X_n]$. But this method is not very efficient. Therefore in the case I principal I propose an alternative algorithm. This approach is similar to the one for factoring polynomials in several variables over an algebraic number field (see Trager, (Tr)).

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