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LOCALLY REGULAR TOROIDS WITH HEXAGONAL FACES

Lajos Szilassi

Mathematician, (b. Szentes, Hungary, 1942.).

Address: University of Szeged, Juhász Gyula Teacher's Training College

H-6725 Szeged, Boldogasszony sgt. 6. , Hungary

E-mail: **szilassi@jgytf.u-szeged.hu**

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Abstract*: We present examples of polyhedra homeomorphic to a torus such that all of their faces are planar polygons with the same number of sides and all of their vertices are incident to the same number of edges. Neither the faces nor the polyhedra are self-intersecting. We call them locally regular toroids. Because of their local combinatorial regularity, as a kind of symmetry, these structures may be attractive to people who find beauty in geometric patterns. We show that for all F, 7≤ F ≤ 12, there exists a regular toroid whose faces are simple hexagons. For all combinatorial types of these we present an example such that two distinct face have no point in common, apart from a vertex or edge they possibly share. We give the combinatorial structures of these polyhedra together with the coordinates of their vertices.*

Keywords: toroidal polyhedra, equivelar polyhedra

A polyhedron is called *toroid*, if it is like a torus in topological sense. In this paper we deal with toroids of genus one. A toroid is *locally regular*, if the number of edges meeting at the vertices is the same, and all of its faces are polygons with equal sides. A polyhedron satisfying these two conditions is usually called now an *equivelar*

polyhedron (Brehm and Wills, 1993); here we use the term *locally regular* in the particular case of toroidal polyhedra. It is a requirement that the polygons must not be self-intersecting, and the polyhedron should not have overarching faces, i.e. two faces must not have common vertices more than two, and if they have two common vertices, then they have common edges as well.

It is easy to see that there are three classes of regular toroids. (Szilassi, 2000) These classes are denoted by the Schläfli symbol $\{p, q\}$ which means that p edges belong to a face and q to a vertex. Thus the classes are: $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$.

The vertex-minimal representative of the class $\{3, 6\}$ is the Császár polyhedron having $(V, E, F) = (7, 21, 14)$. (Császár, 1949-50)

In the class $\{6, 3\}$ the minimal number of faces is 7. This polyhedron was discovered by the author, and it was Martin Gardner who first called it the *Szilassi polyhedron.* (Gardner,1978) A metallic sculpture of it is exhibited in the mathematical museum of Fermat's birth-house.

Figure 2*:* An overarching polyhedron with F= 8 faces: the Schwörbel polyhedron

One can easily construct one polyhedron in this class with each face numbers 9, 12, 15, …; in general, this number is of the form $a * b$, $a \geq 3$, $b \geq 3$ (Grünbaum and Szilassi) However, it is difficult to find examples for all of the various combinatorial types. Since 1988 we have known a toroid with 8 hexagons (Schwörbel, 1988); however, this is an overarching polyhedron, that is, for each of its faces there is another face such that they have two edges in common.

We obtained (in 1977) the toroid with 7 hexagons from the Császár polyhedron by applying to its faces, vertices and edges a reciprocation with respect to a sphere (Szilassi, 1986), (Gailiunas and Sharp, 2005). Since the Császár polyhedron is necessarily non-convex, and the polarity is a projective geometric transformation, in the dual figure self-intersecting faces were produced. To explore these undesirable self-intersections, we used a computer-aided procedure. By varying the vertex positions of this 7-vertex polyhedra we succeeded in eliminating all self-intersections of any faces in its dual. These investigations required several months with the computers of that time; now they can be performed within minutes. A program suitable for this purpose is *Euler 3D* (http://www.euler3d.hu/). This has made possible to find polyhedra in the class {6,3} that were unknown till now. If we are able to find coordinates for a polyhedron with a given number of faces and a given combinatorial structure, such that its dual obtained by a suitable reciprocation is free of self-

intersections, then we can further fine-tune the coordinates with some ”aesthetic” goals in mind. For example, we could aim for some more symmetrical appearance.

In 2008 U. Brehm and W. Kühnel verified, that polyhedra {3,6} having 7, 8, 10 and 11 vertices have only a single combinatorial type. But the polyhedra with 9, 13 and 14 vertices have 2, polyhedra with 12 and 15 vertices have 4 different combinatorial types. Accordingly, in class {6,3} four polyhedra with distinct combinatorial type may exist among the regular toroids having 12 faces.

Now we are going to show, that each of these combinatorial types can be realised as polyhedra having no self-intersection or overarching faces in case of F≤12. Each polyhedron belonging to a given combinatorial type can be realised in a number of ways by properly selecting the vertices.

In the Appendix we give (both by numerical data and by planar maps) the combinatorial types of the polyhedra. We also give one or more realizations of these combinatorial types by the coordinates of the vertices.

Two regular toroids of type {6,3} with the same number of faces are isomorphic if and only if the following condition holds. There is a transformation by which the hexagons of the one planar map are transformed to the hexagons of the other map, and, in particular, sends the parallelogram lattice in the one map to the parallelogram lattice of the other map. The parallelograms of this lattice are given by the hexagons labelled by "1". In this way one can check that there exist those and only those combinatorially distinct structures which we have given here.

Since the vertices of the starting polyhedra of type {3,6} have rational coordinates, and in applying the reciprocation one has to solve systems of linear equations, we obtain toroids of type {6, 3} also with rational coordinates.

By a proper selection of coordinates it is possible to ensure that all polyhedra have congruent pairs of faces. The reader is encouraged to find out the symmetry properties of such polyhedra. For example polyhedron **F10 A1** has no symmetry axis, while polyhedron **F10 A2**, has one, and **F10 A3** and **F10 A4** which belongs to the same combinatorial type, has three symmetry axes. It is to be noted, that the polyhedra specified here have no symmetry plane, except for F**12 C1** and **F12 C2**, so each of them has a "right-handed" and a "left-handed" version. It is suspected that these polyhedra have no variant with mirror symmetry.

It is relatively easy to find examples in class {3,6} for each of the combinatorial types of polyhedra with 13, 14 and 15 vertices. However, a hard unsolved dual problem is the identification of polyhedra with 13 and 14 faces in class {6,3}, or the specification of all the combinatorially different polyhedra having 15 faces.

Acknowledgement: The author would like to thank to the anonymous referee for the valuable comments concerning the global combinatorial regularity of the **F8 A**, **F9 B, F12 C** and **F12 D** maps. (The other maps are not globally regular). This can be seen from the following property of our figures: these four maps are axially symmetric, while the other are not.

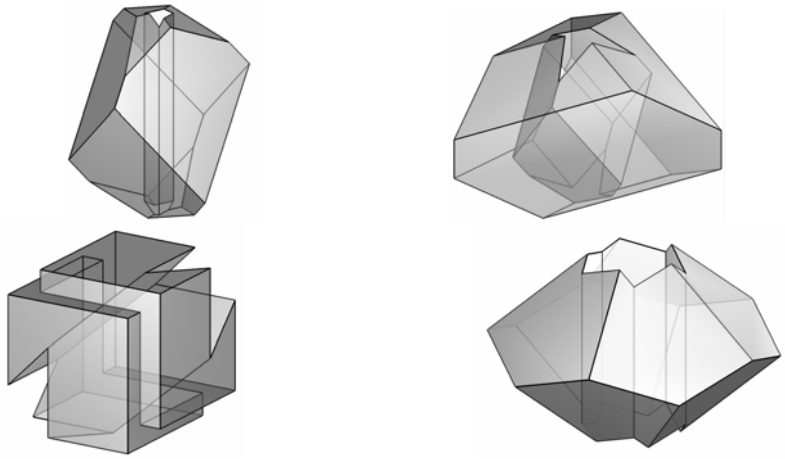

Figure2: Regular toroids with F= 15 faces: the first three are combinatorially isomorphic, but the fourth one is combinatorially different from the other three .(Grünbaum and Szilassi)

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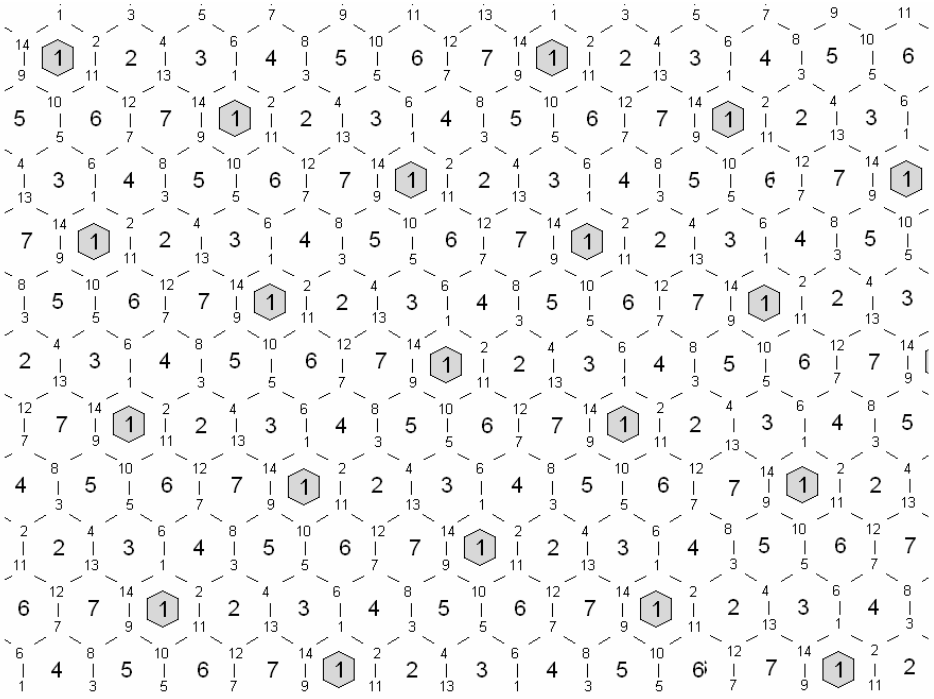
APPENDIX

Toroids with 7 faces

Combinatorial structure:

Faces:	**F7 A**					
1.	*1*	*2*	*11*	*10*	*9*	*14*
2.	*3*	*4*	*13*	*12*	*11*	*2*
3.	*5*	*6*	*1*	*14*	*13*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*



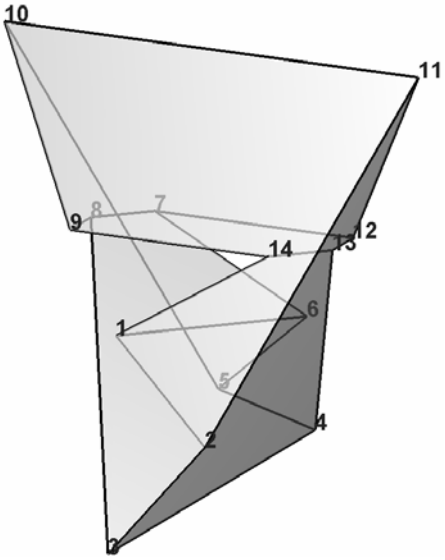
Vertices and realization:

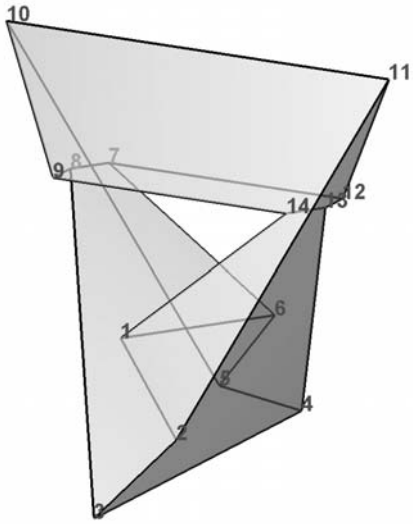
F7 A1

V:	x	y	z
1.	3.75	3.75	-3
2.	-2	5	-8
3.	0	12.6	-12
4.	0	-12.6	-12
5.	2	-5	-8
6.	-3.75	-3.75	-3
7.	4.5	-2.5	2
8.	7	0	2
9.	7	2.5	2
10.	12	0	12
11.	-12	0	12
12.	-7	-2.5	2
13.	-7	0	2
14.	-4.5	2.5	2

F7 A2

V:	x	y	z
1.	8/3	4	-4
2.	-2.5	5.25	-9
3.	0	12	-12
4.	0	-12	-12
5.	1.5	-5.25	-9
6.	-8/3	-4	-4
7.	20/3	-2	4
8.	8	0	4
9.	8	2	4
10.	12	0	12
11.	-12	0	12
12.	-8	-2	4
13.	-8	0	4
14.	-20/3	2	4

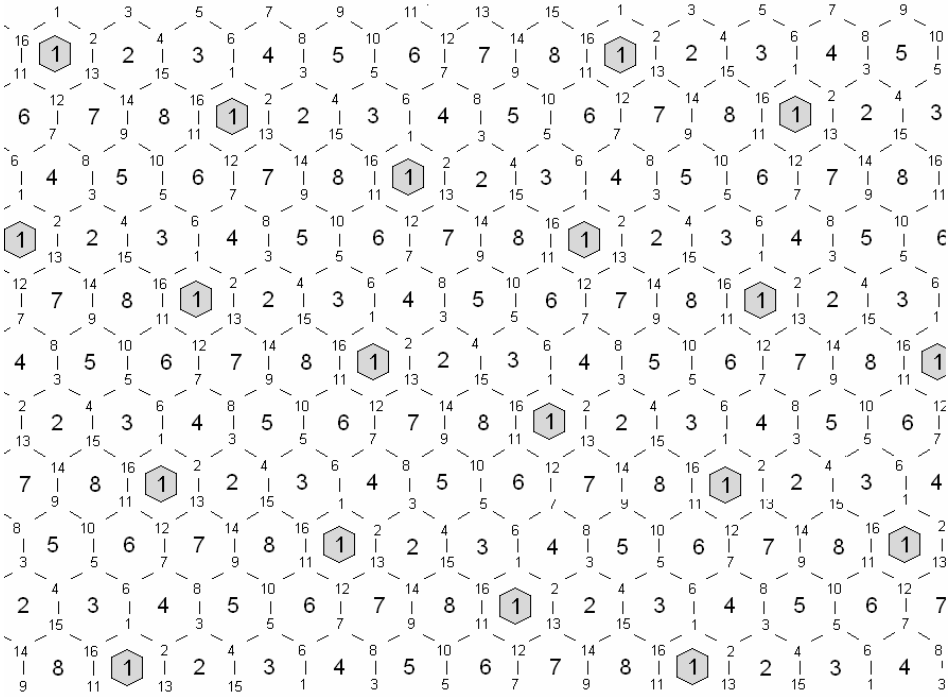




Toroids with 8 faces

Combinatorial structure:

Faces:	**F8 A**					
1.	*1*	*2*	*13*	*12*	*11*	*16*
2.	*3*	*4*	*15*	*14*	*13*	*2*
3.	*5*	*6*	*1*	*16*	*15*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*
8.	*15*	*16*	*11*	*10*	*9*	*14*



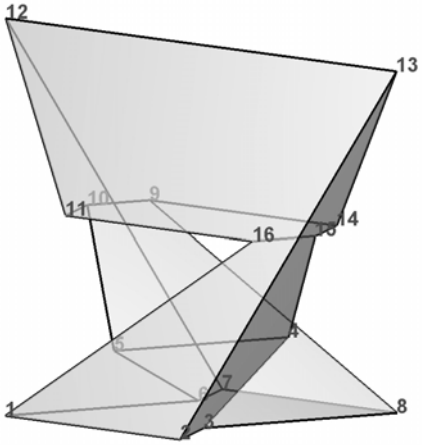
Vertices and realization:

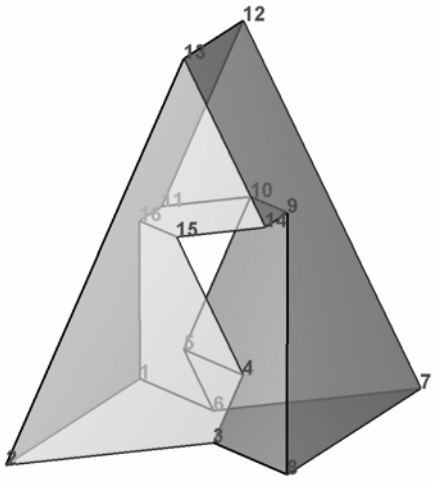
F8 A 1

V:	x	y	z
1.	9.25	5.25	-9
2.	-1.5	5.25	-9
3.	-1.5	2.5	-9
4.	-3.5	-3.5	-5
5.	3.5	3.5	-5
6.	1.5	-2.5	-9
7.	1.5	-5.25	-9
8.	-9.25	-5.25	-9
9.	4.5	-2.5	2
10.	7	0	2
11.	7	2.5	2
12.	12	0	12
13.	-12	0	12
14.	-7	-2.5	2
15.	-7	0	2
16.	-4.5	2.5	2

F8 A2

V:	x	y	z
1.	14	-2	-10
2.	-4	16	-10
3.	-4	2	-10
4.	-4	0	-6
5.	4	0	-6
6.	4	-2	-10
7.	4	-16	-10
8.	-14	2	-10
9.	-1	-4.5	3
10.	4	-4.5	3
11.	4	1.5	3
12.	4	-4	14
13.	-4	4	14
14.	-4	-1.5	3
15.	-4	4.5	3
16.	1	4.5	3

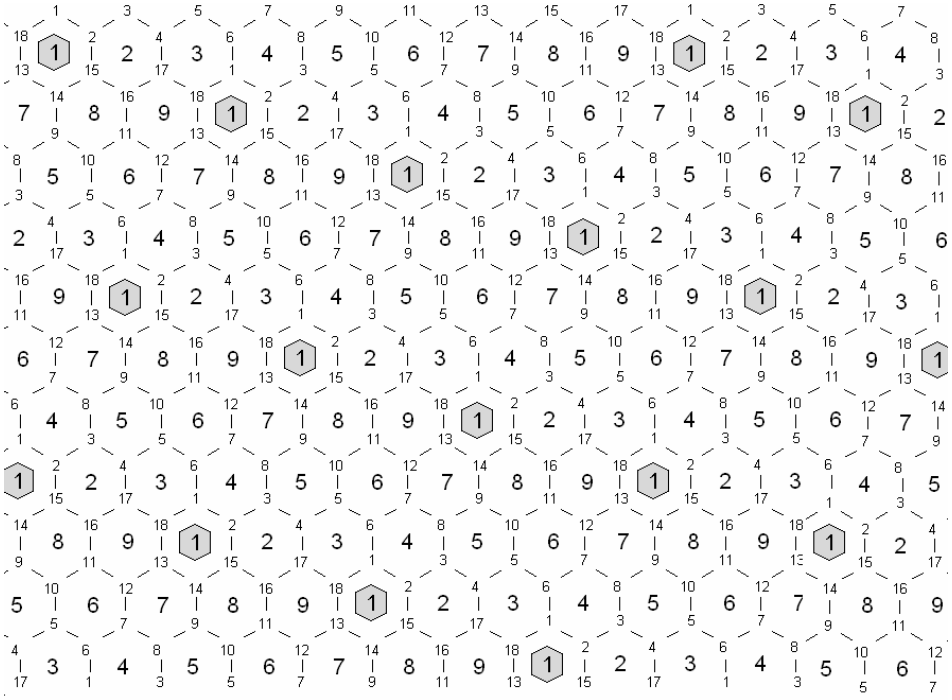




Toroid with 9 faces

Combinatorial structure type **A**:

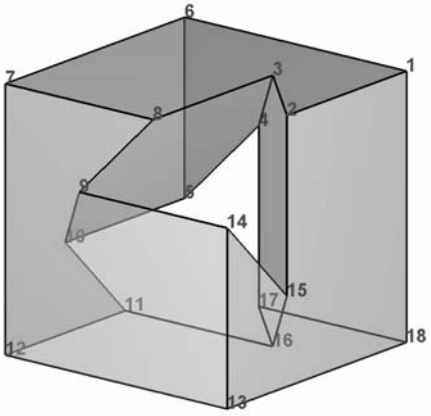
Faces:	**F9 A**					
1.	*1*	*2*	*15*	*14*	*13*	*18*
2.	*3*	*4*	*17*	*16*	*15*	*2*
3.	*5*	*6*	*1*	*18*	*17*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*
8.	*15*	*16*	*11*	*10*	*9*	*14*
9.	*17*	*18*	*13*	*12*	*11*	*16*

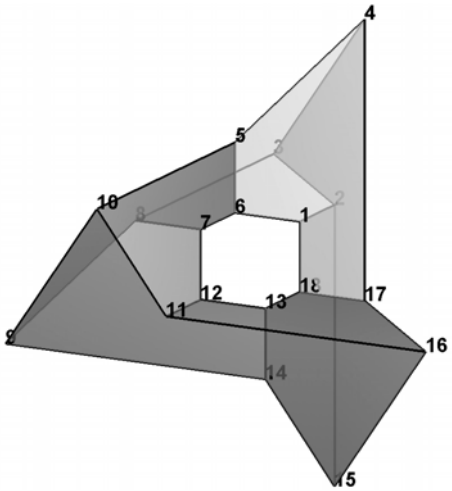


Vertices and realization of the type **A**:

	F9 A1		
V:	x	y	z
1.	-3	-3	3
2.	-3	1	3
3.	-1	-1	3
4.	1	-3	1
5.	3	-3	-1
6.	3	-3	3
7.	3	3	3
8.	-1	3	3
9.	1	3	1
10.	3	1	-1
11.	3	-1	-3
12.	3	3	-3
13.	-3	3	-3
14.	-3	3	1
15.	-3	1	-1
16.	-1	-1	-3
17.	1	-3	-3
18.	-3	-3	-3

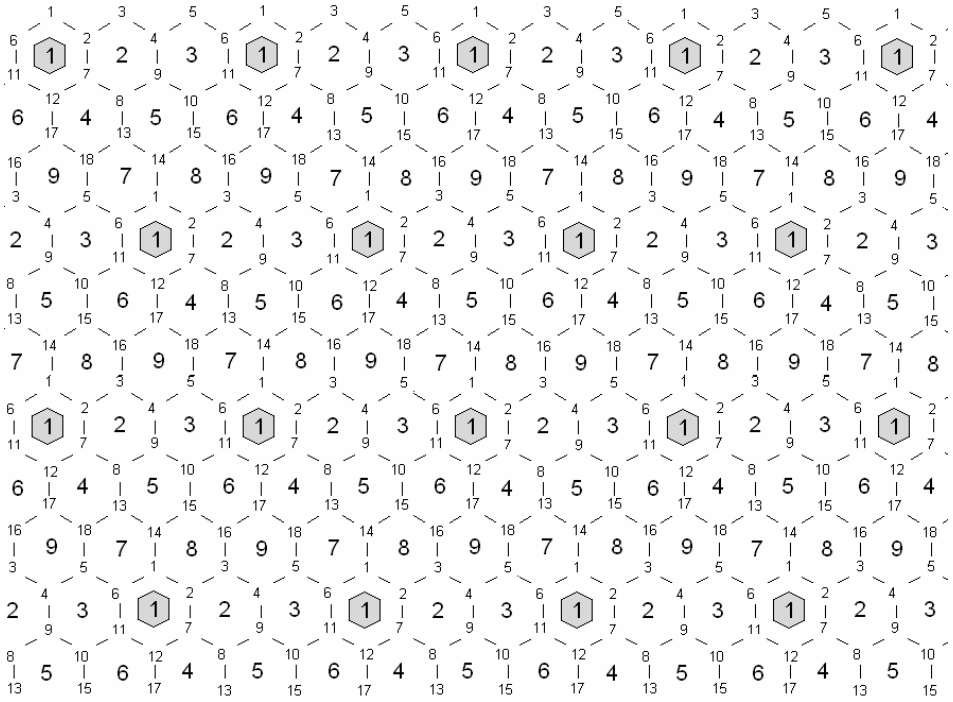
	F9 A2		
V:	x	y	z
1.	-1	-1	1
2.	-1	-3	1
3.	3	-7	1
4.	-3	-1	7
5.	1	-1	3
6.	1	-1	1
7.	1	1	1
8.	3	1	1
9.	7	1	-3
10.	1	7	3
11.	1	3	-1
12.	1	1	-1
13.	-1	1	-1
14.	-1	1	-3
15.	-1	-3	-7
16.	-7	3	-1
17.	-3	-1	-1
18.	-1	-1	-1





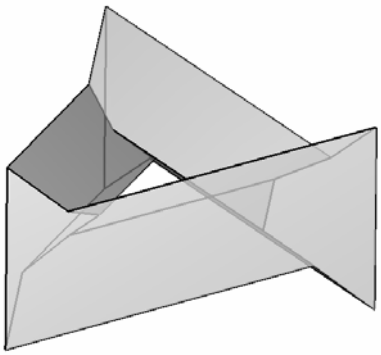
Combinatorial structure type **B**:

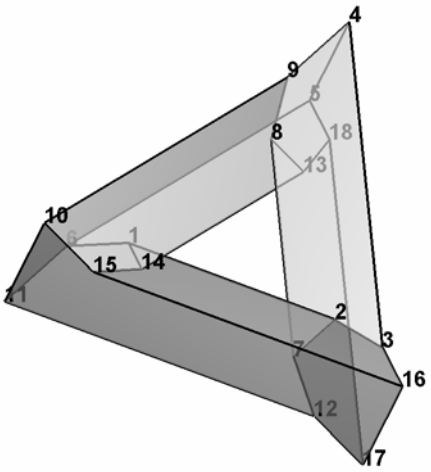
Faces:	**F9 B**					
1.	*1*	*2*	*7*	*12*	*11*	*6*
2.	*3*	*4*	*9*	*8*	*7*	*2*
3.	*5*	*6*	*11*	*10*	*9*	*4*
4.	*7*	*8*	*13*	*18*	*17*	*12*
5.	*9*	*10*	*15*	*14*	*13*	*8*
6.	*11*	*12*	*17*	*16*	*15*	*10*
7.	*13*	*14*	*1*	*6*	*5*	*18*
8.	*15*	*16*	*3*	*2*	*1*	*14*
9.	*17*	*18*	*5*	*4*	*3*	*16*



Vertices and realization of the type **A**:

	F9 B		
V:	**X**	**Y**	**Z**
1.	*15*	*9*	*1*
2.	*-15*	*-1*	*-9*
3.	*-24*	*0*	*-12*
4.	*-9*	*-15*	*33*
5.	*9*	*-33*	*15*
6.	*24*	*12*	*0*
7.	*-9*	*1*	*-15*
8.	*1*	*-9*	*15*
9.	*0*	*-12*	*24*
10.	*15*	*33*	*9*
11.	*33*	*15*	*-9*
12.	*-12*	*0*	*-24*
13.	*-1*	*-15*	*9*
14.	*9*	*15*	*-1*
15.	*12*	*24*	*0*
16.	*-33*	*9*	*-15*
17.	*-15*	*-9*	*-33*
18.	*0*	*-24*	*12*

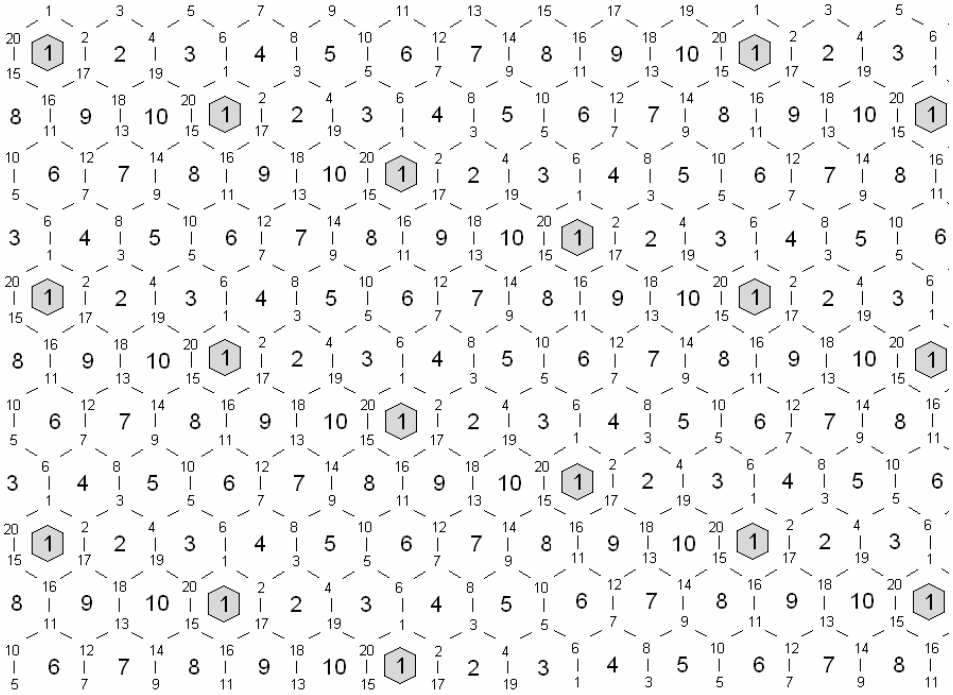




Toroids with 10 faces

Combinatorial structure:

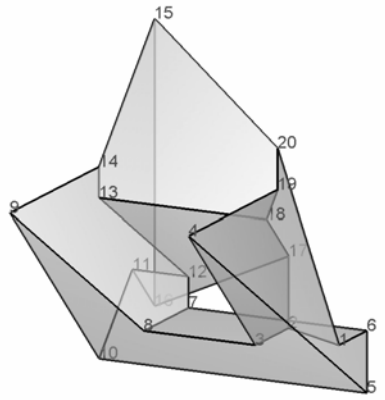
Faces:	**F10 A**					
1.	*1*	*2*	*17*	*16*	*15*	*20*
2.	*3*	*4*	*19*	*18*	*17*	*2*
3.	*5*	*6*	*1*	*20*	*19*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*
8.	*15*	*16*	*11*	*10*	*9*	*14*
9.	*17*	*18*	*13*	*12*	*11*	*16*
10.	*19*	*20*	*15*	*14*	*13*	*18*



Vertices and realization:

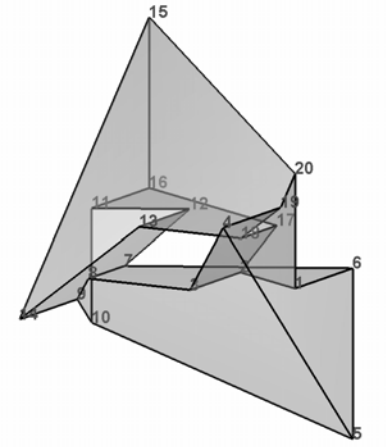
F10A 1

V:	X	Y	Z
1.	-10.5	0.75	-3
2.	-6	-1.5	-3
3.	-6	3	-3
4.	-6	12	6
5.	-12	0	-6
6.	-10.5	-3	-3
7.	1.5	-3	-3
8.	1.5	3	-3
9.	6	12	6
10.	6	0	-6
11.	6	-4.5	-1.5
12.	2.25	-4.5	-1.5
13.	5.25	1.5	4.5
14.	6	0	6
15.	6	-7.5	13.5
16.	6	-7.5	-4.5
17.	-6	-1.5	1.5
18.	-6	1.5	4.5
19.	-6	0	6
20.	-5	-2	8



F10A 2

V:	X	Y	Z
1.	-8	-4	-2
2.	-2	-7	-2
3.	-2	0	-2
4.	-8	6	4
5.	-8	-12	-14
6.	-8	-12	-2
7.	6	-5	-2
8.	6	0	-2
9.	8	-2	-4
10.	8	-4	-6
11.	8	-4	2
12.	2	-7	2
13.	2	0	2
14.	8	6	-4
15.	8	-12	14
16.	8	-12	2
17.	-6	-5	2
18.	-6	0	2
19.	-8	-2	4
20.	-8	-4	6

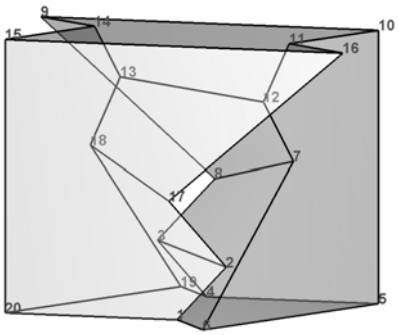


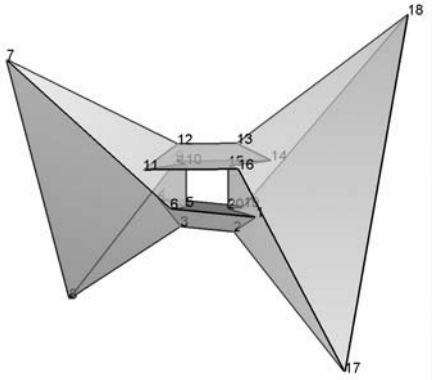
F10A 3

V:	x	y	z
1.	-2.4	6	-6
2.	-4.8	6	-3.6
3.	4.8	-6	-3.6
4.	2.4	-6	-6
5.	-6	-6	-6
6.	-6	10.5	-6
7.	-6	2	0.8
8.	2	-6	-0.8
9.	10.5	-6	6
10.	-6	-6	6
11.	-6	2.4	6
12.	-6	4.8	3.6
13.	6	-4.8	3.6
14.	6	-2.4	6
15.	6	6	6
16.	-10.5	6	6
17.	-2	6	-0.8
18.	6	-2	0.8
19.	6	-10.5	-6
20.	6	6	-6

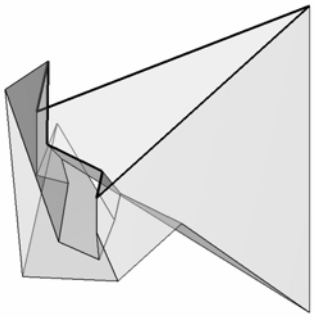
F10A 4

V:	x	y	z
1.	2.5	4	-1.5
2.	0.5	2	-3
3.	-0.5	-2	-3
4.	-2.5	-4	-1.5
5.	0	-1.5	-1.5
6.	0.9	-2.4	-1.5
7.	10.5	-12	10.5
8.	-10.5	-12	-10.5
9.	-0.9	-2.4	1.5
10.	0	-1.5	1.5
11.	2.5	-4	1.5
12.	0.5	-2	3
13.	-0.5	2	3
14.	-2.5	4	1.5
15.	0	1.5	1.5
16.	0.9	2.4	1.5
17.	10.5	12	-10.5
18.	-10.5	12	10.5
19.	-0.9	2.4	-1.5
20.	0	1.5	-1.5

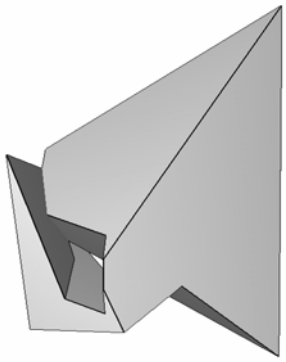




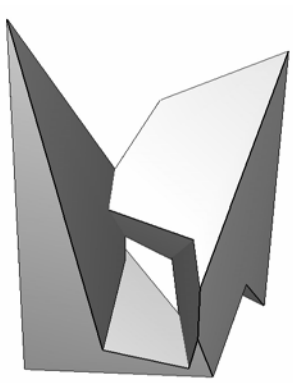
We present some further models of the regular toroids with 10 faces, by which we illustrate the way how we arrived at the results above.

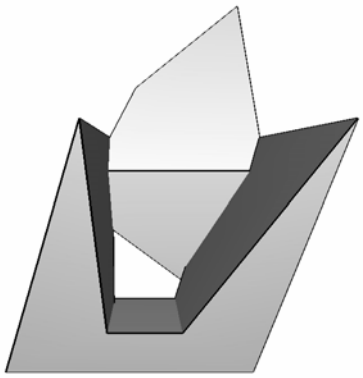


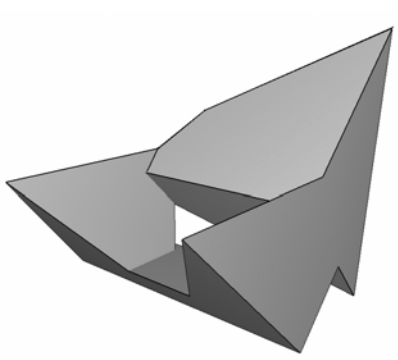
It has a self-intersecting face

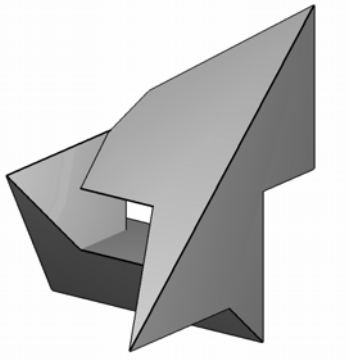


This was the first model without self-intersection that we found.





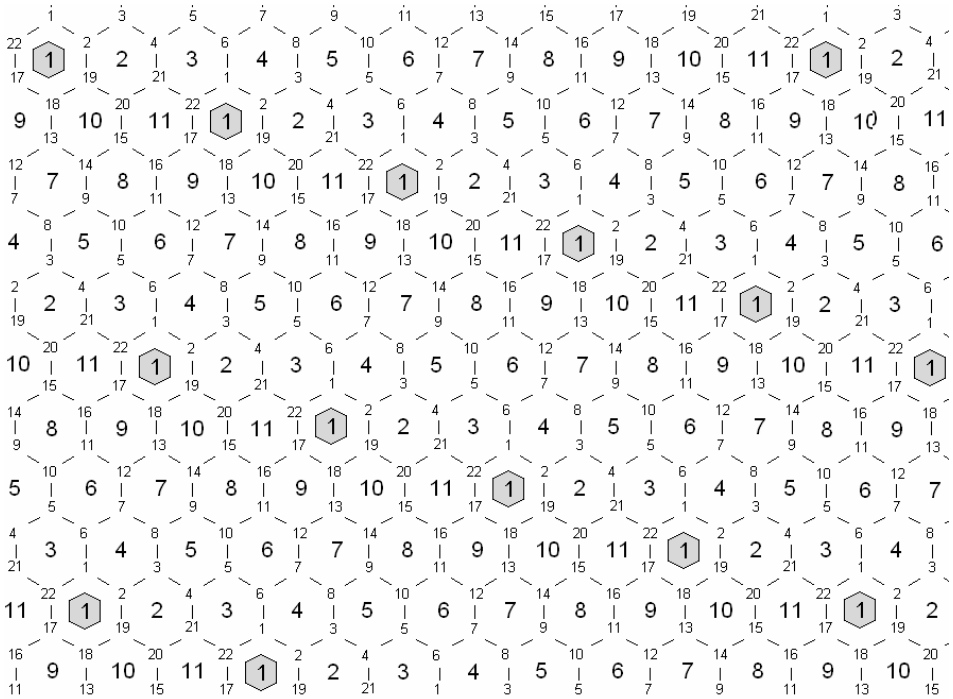




Toroids with 11 faces

Combinatorial structure:

Faces:	**F11 A**					
1.	*1*	*2*	*19*	*18*	*17*	*22*
2.	*3*	*4*	*21*	*20*	*19*	*2*
3.	*5*	*6*	*1*	*22*	*21*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*
8.	*15*	*16*	*11*	*10*	*9*	*14*
9.	*17*	*18*	*13*	*12*	*11*	*16*
10.	*19*	*20*	*15*	*14*	*13*	*18*
11.	*21*	*22*	*17*	*16*	*15*	*20*



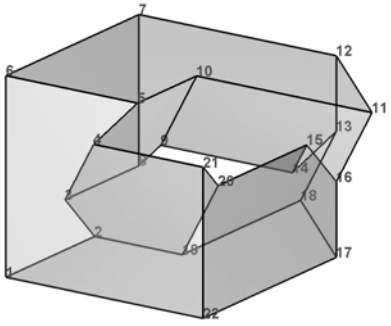
Vertices and realization:

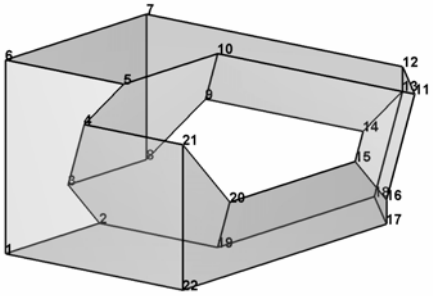
F11 A1

V:	X	Y	Z
1.	6	6	-4
2.	6	0	-4
3.	6	2	-2
4.	2	6	2
5.	0	6	4
6.	6	6	4
7.	6	-3	4
8.	6	-3	-2
9.	5	-3	-1
10.	0	2	4
11.	-8	2	4
12.	-3	-3	4
13.	-3	-3	1
14.	-1	-3	-1
15.	-3	-1	1
16.	-3	-3	-1
17.	-3	-3	-4
18.	2	-8	-4
19.	2	0	-4
20.	-3	5	1
21.	-3	6	2
22.	-3	6	-4

F11 A2

V:	X	Y	Z
1.	6	6	-4
2.	6	0	-4
3.	6	2	-2
4.	2	6	2
5.	0	6	4
6.	6	6	4
7.	6	-3	4
8.	6	-3	-2
9.	3	-3	1
10.	0	0	4
11.	-10	0	4
12.	-7	-3	4
13.	-7	-3	3
14.	-5	-3	1
15.	-3	-5	-1
16.	-3	-7	-3
17.	-3	-7	-4
18.	0	-10	-4
19.	0	0	-4
20.	-3	3	-1
21.	-3	6	2
22.	-3	6	-4

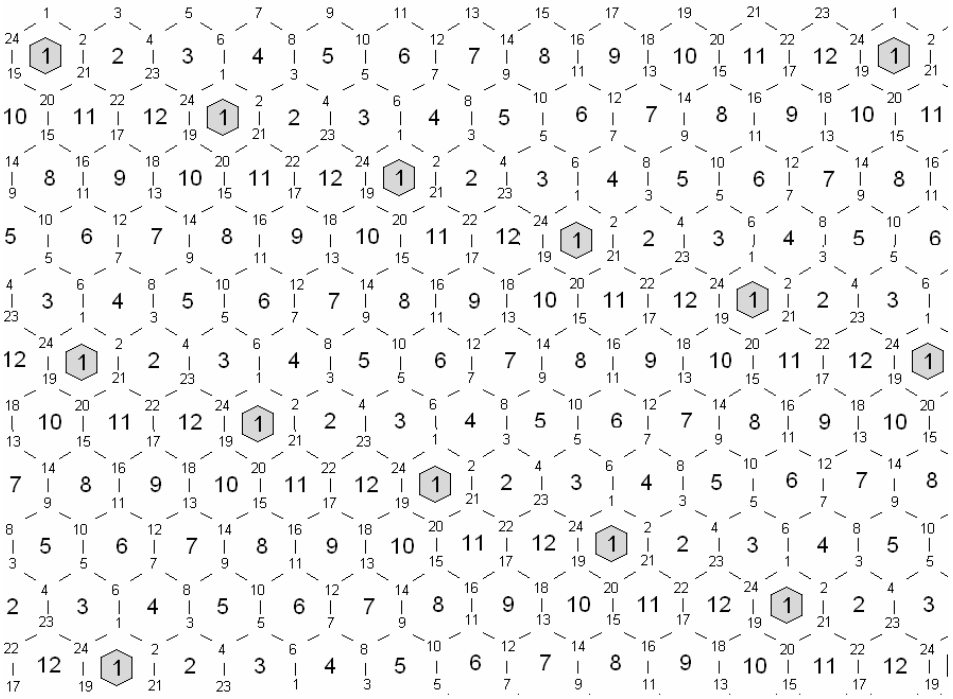




Toroids with 12 faces

Combinatorial structure type **A**:

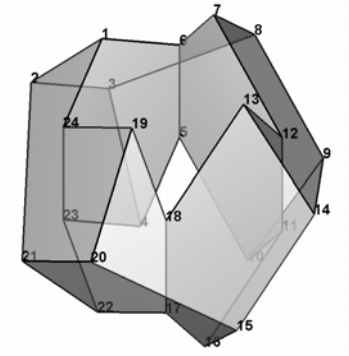
Faces:	**F12 A**					
1.	*1*	*2*	*21*	*20*	*19*	*24*
2.	*3*	*4*	*23*	*22*	*21*	*2*
3.	*5*	*6*	*1*	*24*	*23*	*4*
4.	*7*	*8*	*3*	*2*	*1*	*6*
5.	*9*	*10*	*5*	*4*	*3*	*8*
6.	*11*	*12*	*7*	*6*	*5*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*12*
8.	*15*	*16*	*11*	*10*	*9*	*14*
9.	*17*	*18*	*13*	*12*	*11*	*16*
10.	*19*	*20*	*15*	*14*	*13*	*18*
11.	*21*	*22*	*17*	*16*	*15*	*20*
12.	*23*	*24*	*19*	*18*	*17*	*22*

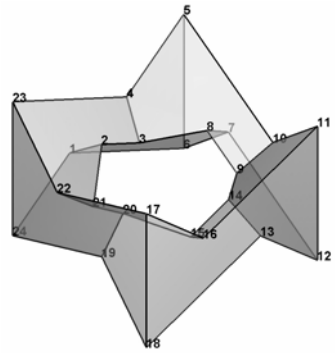


Vertices and realization of the type **A**:

	F12A1		
V:	x	y	z
1.	3	1	7
2.	6	2	2
3.	6	-2	-2
4.	3	-1	-7
5.	3	-3	-3
6.	3	-3	3
7.	1	-3	7
8.	2	-6	2
9.	-2	-6	-2
10.	-1	-3	-7
11.	-3	-3	-3
12.	-3	-3	3
13.	-3	-1	7
14.	-6	-2	2
15.	-6	2	-2
16.	-3	1	-7
17.	-3	3	-3
18.	-3	3	3
19.	-1	3	7
20.	-2	6	2
21.	2	6	-2
22.	1	3	-7
23.	3	3	-3
24.	3	3	3

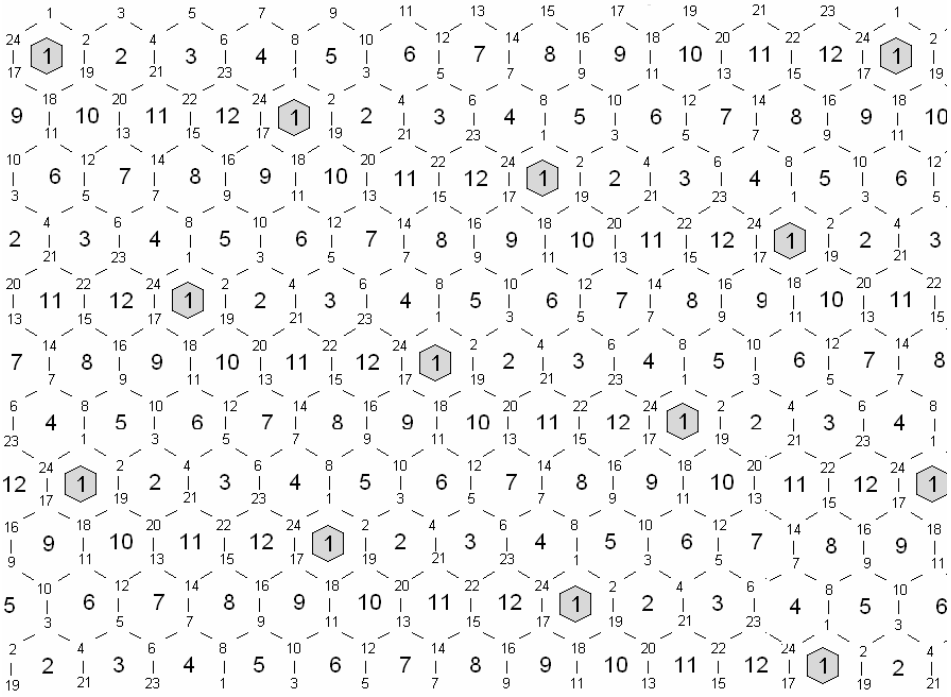
	F12 A2		
V:	x	y	z
1.	6	2	-1
2.	4	4/3	2/3
3.	4	-4/3	- 2/3
4.	6	-2	1
5.	6	-6	5
6.	6	-6	-5
7.	2	-6	-1
8.	4/3	-4	2/3
9.	-4/3	-4	- 2/3
10.	-2	-6	1
11.	-6	-6	5
12.	-6	-6	-5
13.	-6	-2	-1
14.	-4	-4/3	2/3
15.	-4	4/3	- 2/3
16.	-6	2	1
17.	-6	6	5
18.	-6	6	-5
19.	-2	6	-1
20.	-4/3	4	2/3
21.	4/3	4	- 2/3
22.	2	6	1
23.	6	6	5
24.	6	6	-5





Combinatorial structure type **B**:

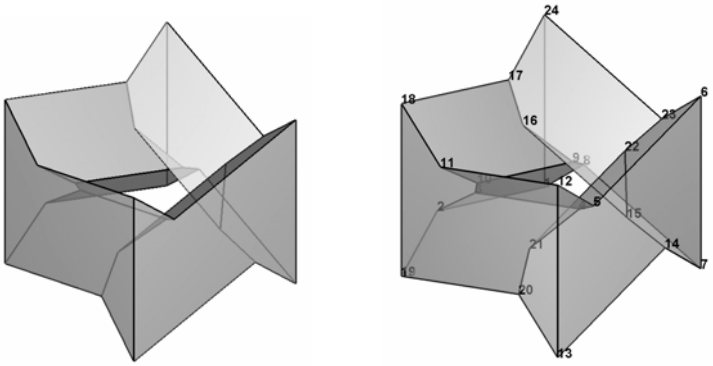
Faces:	**F12 B**					
1.	*1*	*2*	*19*	*18*	*17*	*24*
2.	*3*	*4*	*21*	*20*	*19*	*2*
3.	*5*	*6*	*23*	*22*	*21*	*4*
4.	*7*	*8*	*1*	*24*	*23*	*6*
5.	*9*	*10*	*3*	*2*	*1*	*8*
6.	*11*	*12*	*5*	*4*	*3*	*10*
7.	*13*	*14*	*7*	*6*	*5*	*12*
8.	*15*	*16*	*9*	*8*	*7*	*14*
9.	*17*	*18*	*11*	*10*	*9*	*16*
10.	*19*	*20*	*13*	*12*	*11*	*18*
11.	*21*	*22*	*15*	*14*	*13*	*20*
12.	*23*	*24*	*17*	*16*	*15*	*22*



Vertices and realization of the type **B**:

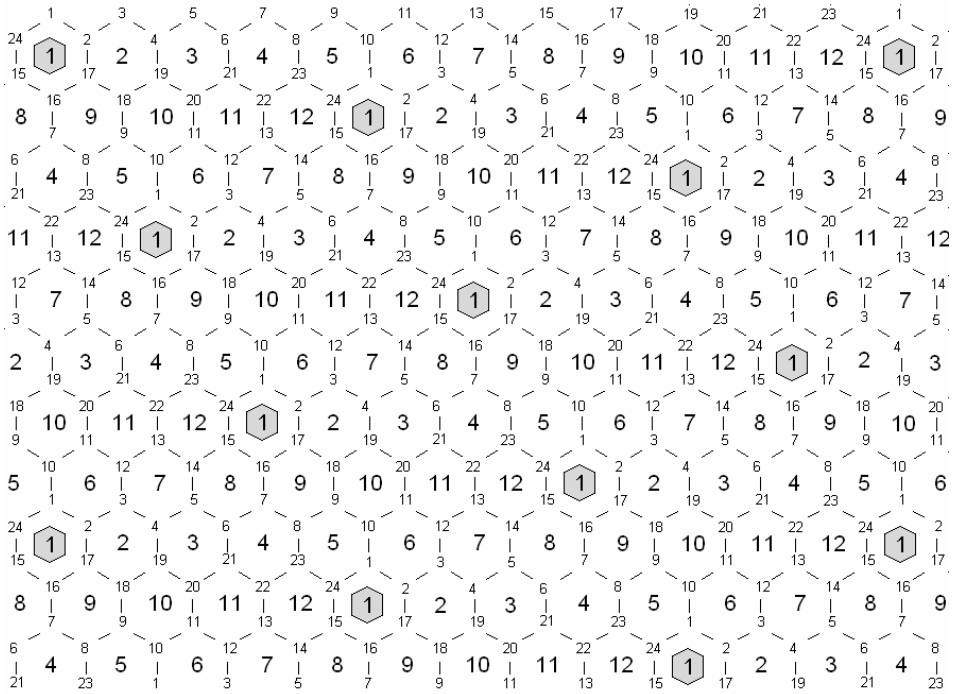
F12 B

V:	X	Y	Z
1.	6	-6	-6
2.	6	3	-3
3.	4	2	-4/3
4.	-4	2	4/3
5.	-6	3	3
6.	-6	-6	6
7.	-6	-6	-6
8.	3	-6	-3
9.	2	-4	-4/3
10.	2	4	4/3
11.	3	6	3
12.	-6	6	6
13.	-6	6	-6
14.	-6	-3	-3
15.	-4	-2	-4/3
16.	4	-2	4/3
17.	6	-3	3
18.	6	6	6
19.	6	6	-6
20.	-3	6	-3
21.	-2	4	-4/3
22.	-2	-4	4/3
23.	-3	-6	3
24.	6	-6	6



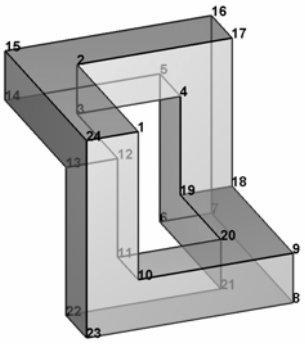
Combinatorial structure type **C**:

Faces:	**F12 C**					
1.	*1*	*2*	*17*	*16*	*15*	*24*
2.	*3*	*4*	*19*	*18*	*17*	*2*
3.	*5*	*6*	*21*	*20*	*19*	*4*
4.	*7*	*8*	*23*	*22*	*21*	*6*
5.	*9*	*10*	*1*	*24*	*23*	*8*
6.	*11*	*12*	*3*	*2*	*1*	*10*
7.	*13*	*14*	*5*	*4*	*3*	*12*
8.	*15*	*16*	*7*	*6*	*5*	*14*
9.	*17*	*18*	*9*	*8*	*7*	*16*
10.	*19*	*20*	*11*	*10*	*9*	*18*
11.	*21*	*22*	*13*	*12*	*11*	*20*
12.	*23*	*24*	*15*	*14*	*13*	*22*

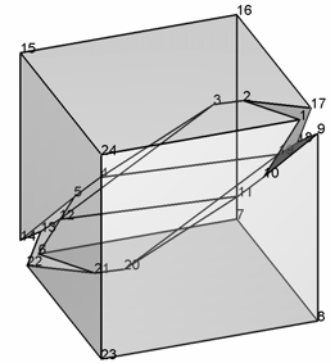


Vertices and realization of the type **C**:

	F12 C1		
V:	X	Y	Z
1.	3	6	6
2.	3	-3	6
3.	3	-3	3
4.	-3	-3	3
5.	-3	-6	3
6.	-3	-6	-6
7.	-6	-6	-6
8.	-6	6	-6
9.	-6	6	-3
10.	3	6	-3
11.	3	3	-3
12.	3	3	3
13.	6	3	3
14.	6	-6	3
15.	6	-6	6
16.	-6	-6	6
17.	-6	-3	6
18.	-6	-3	-3
19.	-3	-3	-3
20.	-3	3	-3
21.	-3	3	-6
22.	6	3	-6
23.	6	6	-6
24.	6	6	6

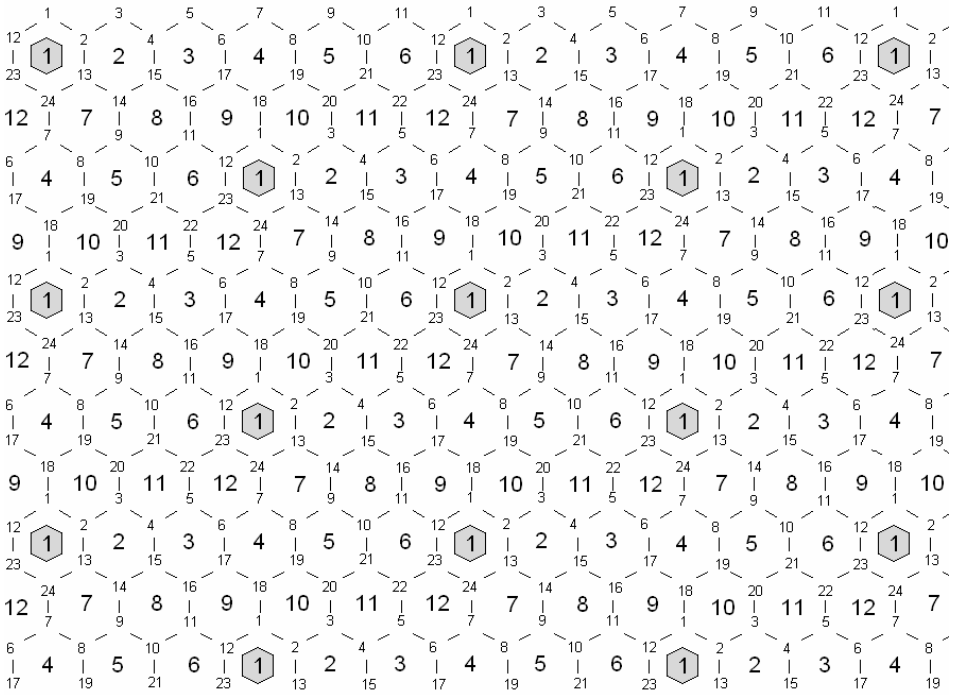


	F12 C2		
V:	X	Y	Z
1.	-5	6	6
2.	-3	3	6
3.	-1.8	1.8	5.4
4.	1.8	-5.4	-1.8
5.	3	-6	-3
6.	5	-6	-6
7.	-6	-6	-6
8.	-6	6	-6
9.	-6	6	5
10.	-3	6	3
11.	-1.8	5.4	1.8
12.	5.4	-1.8	-1.8
13.	6	-3	-3
14.	6	-6	-5
15.	6	-6	6
16.	-6	-6	6
17.	-6	5	6
18.	-6	3	3
19.	-5.4	1.8	1.8
20.	1.8	-1.8	-5.4
21.	3	-3	-6
22.	6	-5	-6
23.	6	6	-6
24.	6	6	6



Combinatorial structure type **D**:

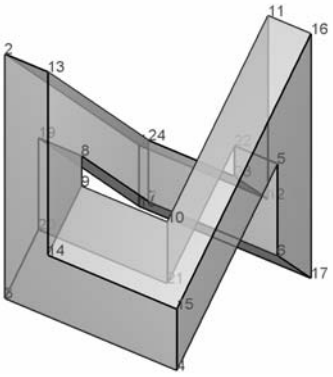
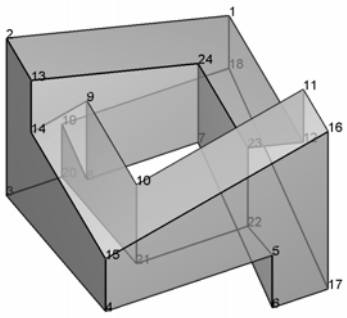
Faces:	**F12 D**					
1.	*1*	*2*	*13*	*24*	*23*	*12*
2.	*3*	*4*	*15*	*14*	*13*	*2*
3.	*5*	*6*	*17*	*16*	*15*	*4*
4.	*7*	*8*	*19*	*18*	*17*	*6*
5.	*9*	*10*	*21*	*20*	*19*	*8*
6.	*11*	*12*	*23*	*22*	*21*	*10*
7.	*13*	*14*	*9*	*8*	*7*	*24*
8.	*15*	*16*	*11*	*10*	*9*	*14*
9.	*17*	*18*	*1*	*12*	*11*	*16*
10.	*19*	*20*	*3*	*2*	*1*	*18*
11.	*21*	*22*	*5*	*4*	*3*	*20*
12.	*23*	*24*	*7*	*6*	*5*	*22*



Vertices and realization of the type **D**:

F12D 1

V:	X	Y	Z
1.	-6	-6	-4
2.	6	-6	8
3.	6	-6	-8
4.	6	6	-8
5.	-3	6	1
6.	-3	6	-5
7.	-3	-3	-5
8.	3	-3	1
9.	3	-3	-1
10.	3	3	-1
11.	-6	3	8
12.	-6	3	-4
13.	6	-3	8
14.	6	-3	-4
15.	6	6	-4
16.	-6	6	8
17.	-6	6	-8
18.	-6	-6	-8
19.	3	-6	1
20.	3	-6	-5
21.	3	3	-5
22.	-3	3	1
23.	-3	3	-1
24.	-3	-3	-

F12D 2

V:	X	Y	Z
1.	-6	-6	2
2.	6	-6	6
3.	6	-6	-6
4.	6	6	-2
5.	-3	6	-2
6.	-3	6	-6
7.	-3	-3	-3
8.	3	-3	-3
9.	3	-3	3
10.	3	3	3
11.	-6	3	6
12.	-6	3	2
13.	6	-3	6
14.	6	-3	2
15.	6	6	2
16.	-6	6	6
17.	-6	6	-6
18.	-6	-6	-2
19.	3	-6	-2
20.	3	-6	-6
21.	3	3	-3
22.	-3	3	-3
23.	-3	3	3
24.	-3	-3	3