GCLC Prover Output for conjecture “thm-Ceva”

Groebner bases method used

July 10, 2009

1 Construction and prover internal state

Construction commands:

- Point A
- Point B
- Point C
- Point P
- Line a: B C
- Line b: A C
- Line c: A B
- Line pa: P A
- Line pb: P B
- Line pc: P C
- Intersection of lines, D: a pa
- Intersection of lines, E: b pb
- Intersection of lines, F: c pc

Coordinates assigned to the points:

- $A = (0, 0)$
- $B = (u_1, 0)$
- $C = (u_2, u_3)$
- $P = (u_4, u_5)$
- $D = (x_2, x_1)$
- $E = (x_4, x_3)$
- $F = (x_6, 0)$

Conjecture(s):

1. Given conjecture

    - **GCLC code:**

      ```
      equal { mult { mult { sratio A F F B } { sratio B D D C } } { sratio C E E A } }
      ```

    - **Expression:**
      $\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1$
    - **Expression after rationalization:**
      $(((x(A) - x(F)) \cdot (y(B) - y(D))) \cdot (y(C) - y(E))) = (((1 \cdot (x(F) - x(B))) \cdot (y(D) - y(C))) \cdot (y(E) - y(A)))$

2 Resolving constructed lines

- $a \ni B, C, D$
- $b \ni A, C, E$
- $c \ni A, B, F$; line is horizontal (i.e., $y(A) = y(B)$)
- $pa \ni P, A, D$
- $pb \ni P, B, E$
- $pc \ni P, C, F$

3 Creating polynomials from hypotheses

- Point A
  no condition
- Point B
  no condition
- Point C
  no condition
- Point P
  no condition
- Line a: B C
  - point B is on the line (B, C)
    no condition
  - point C is on the line (B, C)
    no condition

- Line b: A C
  - point A is on the line (A, C)
    no condition
  - point C is on the line (A, C)
    no condition
- Line c: A B
  - point A is on the line (A, B)
    no condition
  - point B is on the line (A, B)
    no condition
- Line pa: P A
  - point P is on the line (P, A)
    no condition
  - point A is on the line (P, A)
    no condition
- Line pb: P B
  - point P is on the line (P, B)
    no condition
  - point B is on the line (P, B)
    no condition
- Line pc: P C
  - point P is on the line (P, C)
    no condition
  - point C is on the line (P, C)
    no condition
- Intersection of lines, D: a pa
  - point D is on the line (B, C)

$$p_{70} = -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1$$

  - point D is on the line (P, A)

$$p_{71} = u_5x_2 - u_4x_1$$

- Intersection of lines, E: b pb
  - point E is on the line (A, C)

$$p_{72} = -u_3x_4 + u_2x_3$$

– point E is on the line (P, B)

$$p_{73} = u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1$$

- Intersection of lines, F: *c pc*
  - point F is on the line (A, B) — true by the construction
    no condition
  - point F is on the line (P, C)

$$p_{74} = (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3)$$

4 Creating polynomial from the conjecture

- Processing given conjecture(s).

Conjecture 1:

$$p_{75} = -2x_6x_3x_1 + u_3x_6x_3 + u_3x_6x_1 + u_1x_3x_1 - u_3u_1x_3$$

5 Invoking the theorem prover

The used proving method is Buchberger's method.
Input polynomial system is:

$$\begin{array}{rcl} p_0 & = & -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1 \\ p_1 & = & u_5x_2 - u_4x_1 \\ p_2 & = & -u_3x_4 + u_2x_3 \\ p_3 & = & u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1 \\ p_4 & = & (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3) \end{array}$$

5.1 Iteration 1

Current set is $S_1 =$

$$\begin{array}{rcl} p_0 & = & -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1 \\ p_1 & = & u_5x_2 - u_4x_1 \\ p_2 & = & -u_3x_4 + u_2x_3 \\ p_3 & = & u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1 \\ p_4 & = & (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3) \end{array}$$

1. Creating S-polynomial from the pair (p_0, p_1).
   Forming S-pol of p_0 and p_1:

$$p_{86} = (u_5u_2 - u_5u_1 - u_4u_3)x_1 + u_5u_3u_1$$

   S-pol added.

2. Creating S-polynomial from the pair (p_0, p_2).
   Skipping pair p_0 and p_2 because gcd of their leading monoms is zero.
3. Creating S-polynomial from the pair (p_0, p_3).
   Skipping pair p_0 and p_3 because gcd of their leading monoms is zero.
4. Creating S-polynomial from the pair (p_0, p_4).
   Skipping pair p_0 and p_4 because gcd of their leading monoms is zero.
5. Creating S-polynomial from the pair (p_1, p_2).
   Skipping pair p_1 and p_2 because gcd of their leading monoms is zero.
6. Creating S-polynomial from the pair (p_1, p_3).
   Skipping pair p_1 and p_3 because gcd of their leading monoms is zero.
7. Creating S-polynomial from the pair (p_1, p_4).
   Skipping pair p_1 and p_4 because gcd of their leading monoms is zero.
8. Creating S-polynomial from the pair (p_2, p_3).
   Forming S-pol of p_2 and p_3:
   $$p_{87} = (u_5u_2 - u_4u_3 + u_3u_1)x_3 - u_5u_3u_1$$
   S-pol added.
9. Creating S-polynomial from the pair (p_2, p_4).
   Skipping pair p_2 and p_4 because gcd of their leading monoms is zero.
10. Creating S-polynomial from the pair (p_3, p_4).
    Skipping pair p_3 and p_4 because gcd of their leading monoms is zero.

5.2 Iteration 2

Current set is $S_2 =$

$$\begin{aligned}
p_0 &= -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1 \\
p_1 &= u_5x_2 - u_4x_1 \\
p_2 &= -u_3x_4 + u_2x_3 \\
p_3 &= u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1 \\
p_4 &= (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3) \\
p_5 &= (u_5u_2 - u_5u_1 - u_4u_3)x_1 + u_5u_3u_1 \\
p_6 &= (u_5u_2 - u_4u_3 + u_3u_1)x_3 - u_5u_3u_1
\end{aligned}$$

1. Creating S-polynomial from the pair (p_0, p_5).
   Skipping pair p_0 and p_5 because gcd of their leading monoms is zero.
2. Creating S-polynomial from the pair (p_0, p_6).
   Skipping pair p_0 and p_6 because gcd of their leading monoms is zero.

3. Creating S-polynomial from the pair (p_1, p_5).
   Skipping pair p_1 and p_5 because gcd of their leading monoms is zero.
4. Creating S-polynomial from the pair (p_1, p_6).
   Skipping pair p_1 and p_6 because gcd of their leading monoms is zero.
5. Creating S-polynomial from the pair (p_2, p_5).
   Skipping pair p_2 and p_5 because gcd of their leading monoms is zero.
6. Creating S-polynomial from the pair (p_2, p_6).
   Skipping pair p_2 and p_6 because gcd of their leading monoms is zero.
7. Creating S-polynomial from the pair (p_3, p_5).
   Skipping pair p_3 and p_5 because gcd of their leading monoms is zero.
8. Creating S-polynomial from the pair (p_3, p_6).
   Skipping pair p_3 and p_6 because gcd of their leading monoms is zero.
9. Creating S-polynomial from the pair (p_4, p_5).
   Skipping pair p_4 and p_5 because gcd of their leading monoms is zero.
10. Creating S-polynomial from the pair (p_4, p_6).
   Skipping pair p_4 and p_6 because gcd of their leading monoms is zero.
11. Creating S-polynomial from the pair (p_5, p_6).
   Skipping pair p_5 and p_6 because gcd of their leading monoms is zero.

5.3 Groebner Basis

Groebner basis has 7 polynomials:

$$\begin{aligned}
p_0 &= -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1 \\
p_1 &= u_5x_2 - u_4x_1 \\
p_2 &= -u_3x_4 + u_2x_3 \\
p_3 &= u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1 \\
p_4 &= (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3) \\
p_5 &= (u_5u_2 - u_5u_1 - u_4u_3)x_1 + u_5u_3u_1 \\
p_6 &= (u_5u_2 - u_4u_3 + u_3u_1)x_3 - u_5u_3u_1
\end{aligned}$$

Groebner basis succesfully computed.

6 Reducing Polynomial Conjecture

Reducing with polynomial p_4, the result is:

$$p_{102} = (u_5u_3 - u_3^2)x_6x_3 + (u_5u_3 - u_3^2)x_6x_1 +$$

$$(-2u_5u_2 + u_5u_1 + 2u_4u_3 - u_3u_1)x_3x_1 + (-u_5u_3u_1 + u_3^2u_1)x_3$$

Reducing with polynomial p_4, the result is:

$$p_{103} = (u_5^2u_3 - 2u_5u_3^2 + u_3^3)x_6x_1 + (-2u_5^2u_2 + u_5^2u_1 + 2u_5u_4u_3 + 2u_5u_3u_2 - 2u_5u_3u_1 - 2u_4u_3^2 + u_3^2u_1)x_3x_1 + (u_5^2u_3u_2 - u_5^2u_3u_1 - u_5u_4u_3^2 - u_5u_3^2u_2 + 2u_5u_3^2u_1 + u_4u_3^3 - u_3^3u_1)x_3$$

Reducing with polynomial p_4, the result is:

$$p_{104} = (-2u_5^3u_2 + u_5^3u_1 + 2u_5^2u_4u_3 + 4u_5^2u_3u_2 - 3u_5^2u_3u_1 - 4u_5u_4u_3^2 - 2u_5u_3^2u_2 + 3u_5u_3^2u_1 + 2u_4u_3^3 - u_3^3u_1)x_3x_1 + (u_5^3u_3u_2 - u_5^3u_3u_1 - u_5^2u_4u_3^2 - 2u_5^2u_3^2u_2 + 3u_5^2u_3^2u_1 + 2u_5u_4u_3^3 + u_5u_3^3u_2 - 3u_5u_3^3u_1 - u_4u_3^4 + u_3^4u_1)x_3 + (u_5^3u_3u_2 - u_5^2u_4u_3^2 - 2u_5^2u_3^2u_2 + 2u_5u_4u_3^3 + u_5u_3^3u_2 - u_4u_3^4)x_1$$

Reducing with polynomial p_5, the result is:

$$p_{105} = (u_5^4u_3u_2^2 - 2u_5^3u_4u_3^2u_2 - 2u_5^3u_3^2u_2^2 + u_5^3u_3^2u_2u_1 + u_5^2u_4^2u_3^3 + 4u_5^2u_4u_3^3u_2 - u_5^2u_4u_3^3u_1 + u_5^2u_3^3u_2^2 - 2u_5^2u_3^3u_2u_1 - 2u_5u_4^2u_3^4 - 2u_5u_4u_3^4u_2 + 2u_5u_4u_3^4u_1 + u_5u_3^4u_2u_1 + u_4^2u_3^5 - u_4u_3^5u_1)x_3 + (u_5^4u_3u_2^2 - u_5^4u_3u_2u_1 - 2u_5^3u_4u_3^2u_2 + u_5^3u_4u_3^2u_1 - 2u_5^3u_3^2u_2^2 + 2u_5^3u_3^2u_2u_1 + u_5^2u_4^2u_3^3 + 4u_5^2u_4u_3^3u_2 - 2u_5^2u_4u_3^3u_1 + u_5^2u_3^3u_2^2 - u_5^2u_3^3u_2u_1 - 2u_5u_4^2u_3^4 - 2u_5u_4u_3^4u_2 + u_5u_4u_3^4u_1 + u_4^2u_3^5)x_1$$

Reducing with polynomial p_6, the result is:

Polynomial too big for output (text size is 1442 characters, number of terms is 2)

Reducing with polynomial p_5, the result is:

$$p_{106} = 0$$

Conclusion is reduced to zero.

7 Prover report

Status: The conjecture has been proved.

Space Complexity: The biggest polynomial obtained during proof process contained 45 terms.

Time Complexity: Time spent by the prover is 0.032 seconds.

NDG conditions are:

- $P_{FBF} \neq 0$ i.e., points F and B are not identical (conjecture based assumption).
- $P_{DCD} \neq 0$ i.e., points D and C are not identical (conjecture based assumption).
- $P_{EAE} \neq 0$ i.e., points E and A are not identical (conjecture based assumption).