A q-Nekrasov–Okounkov formula for type $\widetilde{C}$

AEC
Wien
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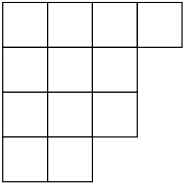
07/07/2022

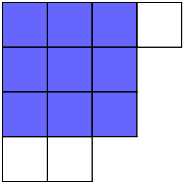
Ferrers diagram and hooks of partitions



$$|\lambda| = 4 + 3 + 3 + 2 = 12$$

$$(4, 3, 3, 2) \in \mathcal{P}$$

Ferrers diagram and hooks of partitions



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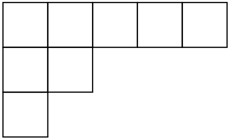
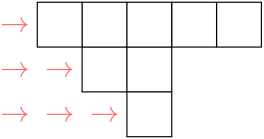
Durfee square

Ferrers diagram and hooks of partitions



$$(5, 2, 1) \in \mathcal{D}$$



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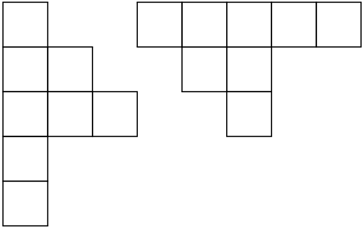
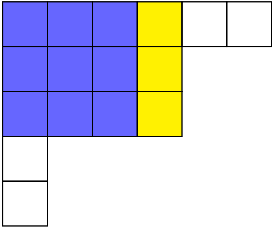
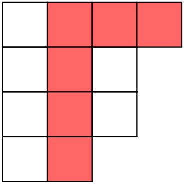


Twice $(5, 2, 1) \in \mathcal{D}$



$(6, 4, 4, 1, 1) \in \mathcal{DD}$

Ferrers diagram and hooks of partitions



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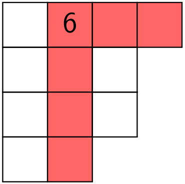

$$(4, 3, 3, 2) \in \mathcal{P}$$

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1		

$(4, 3, 3, 2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) :=$ {hook-lengths}

Ferrers diagram and hooks of partitions

7	**6**	4	1
5	4	2	
4	**3**	1	
2	1	$\mathcal{H}_3$	

$(4, 3, 3, 2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) :=$ {hook-lengths}
- for $t \in \mathbb{N}^*$, $\mathcal{H}_{\mathbf{t}}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$

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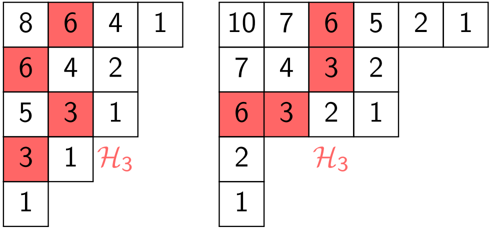

$(4,3,3,2,1) \in \mathcal{P}$ $\quad$ $(6,4,4,1,1) \in \mathcal{DD}$

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Ferrers diagram and hooks of partitions

10	7	6	5	2	1
7	4	3	2		
6	3	2	1		
2		Δ			
1					

(a) $(6,4,4,1,1) \in \mathcal{DD}$

$+$	$+$	$+$	$+$	$+$	$+$
$-$	$+$	$+$	$+$		
$-$	$-$	$+$	$+$		
$-$		Δ			
$-$					

(b) ε_s

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- $\varepsilon_s = \begin{cases} -1 \text{ if } s \text{ is a box strictly below } \Delta \\ 1 \text{ otherwise} \end{cases}$

A q-Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z^2}{h^2}\right) = (T;T)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

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Note $z = t \in \mathbb{N}^* \Rightarrow \lambda$ are t-cores.

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Dehaye–Han (2011), Rains–Warnaar (2018), Carlsson–Rodriguez-Villegas (2018)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2}$$

$$= \prod_{k,r \geq 1} \frac{(1-uq^rT^k)^r(1-u^{-1}q^rT^k)^r}{(1-q^{r-1}T^k)^r(1-q^{r+1}T^k)^r}$$

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Special case $u = q^z$ and $q \to 1$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$.

1. **Partitions**:

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

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2. **Double distinct partitions**:
   (a) for t odd:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-1)/2}$$

   (b) for t even:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}, \kappa) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t/2-1)} \times \mathcal{SC}$$

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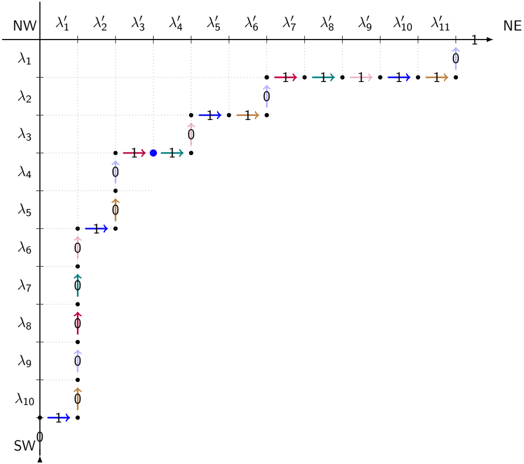
Tools: $\lambda \longleftrightarrow s(\lambda)$ bi-infinite word of 0's and 1's

An example for $\omega \in \mathcal{DD}_{(6)}$

$\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1)$ and its binary correspondence



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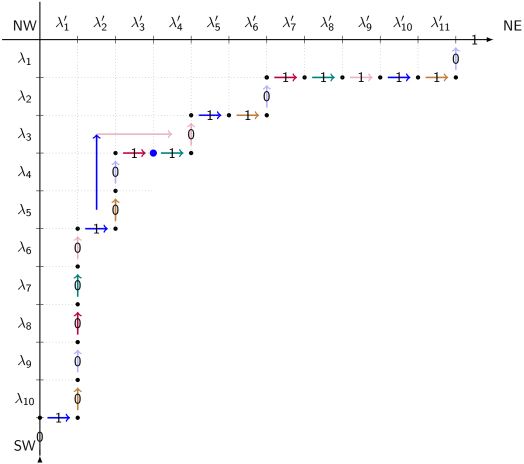

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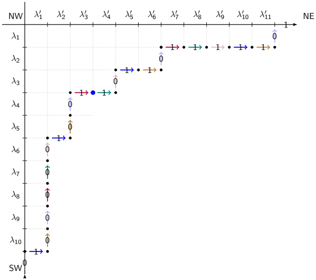

$$s(w_0) = \cdots 000|111 \cdots$$
$$s(w_1) = \cdots 000|011 \cdots$$
$$s(w_2) = \cdots 011|111 \cdots$$
$$s(w_3) = \cdots 000|111 \cdots$$
$$s(w_4) = \cdots 000|001 \cdots$$
$$s(w_5) = \cdots 001|111 \cdots$$

$$s(\omega) = \cdots 0000001000001001|10110111110111111111 \cdots$$

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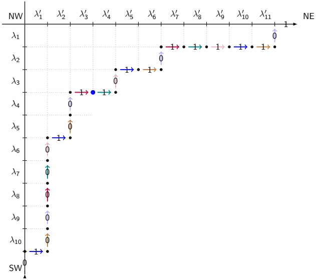

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Garvan–Kim–Stanton (1990):
$\omega \in \mathcal{P}_{(t)} \longleftrightarrow (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t$ such that $\sum_{i=0}^{t-1} n_i = 0$. Here

$$(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6$$

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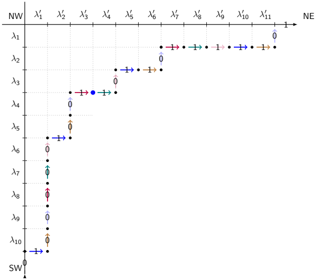

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$$\omega \in \mathcal{DD}_{(6)} \longleftrightarrow (1, -2) \in \mathbb{Z}^2$$

t-cores and vectors of integers

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A vector of integers $(v_1, \dots, v_t) \in \mathbb{Z}^t$ is called a V_t-coding if:

1. $\#\{v_i - i \pmod{2t+2}, i = 1, \dots, t\} = t$
2. $v_i \not\equiv 0 \pmod{2t+2}$, $v_i \not\equiv t+1 \pmod{2t+2}$
3. $0 < v_1 < v_2 < \cdots < v_t$

Hook-length formulas for $\mathcal{DD}_{(2t+2)}$

Theorem [W., 2022]

Set t a strictly positive integer and let τ be a function defined over $\mathbb{Z}$. Then there is a bijection $\phi_t : \omega \mapsto (v_0, \cdots, v_{t-1})$ from $\mathcal{DD}_{(2t+2)}$ to V_t-codings such that:

$$\frac{|\omega|}{2} = \frac{1}{4(t+1)} \sum_{i=1}^{t} v_i^2 - \frac{t(2t+1)}{24}$$

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And, for $\beta_i(\omega) := \#\{h \in \mathcal{H}(\omega) \mid h = 2t+2-i, \varepsilon_h = -1\}$

$$\prod_{\substack{s\in\omega \\ h_s\in\mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s(2t+2))}{\tau(h_s)} = \prod_{i=1}^{2t+1} \left(\frac{\tau(-i)}{\tau(2t+2-i)}\right)^{\beta_i(\omega)}$$

$$\times \prod_{i=1}^{t} \frac{\tau(v_i)}{\tau(i)} \prod_{1\le i<j\le t} \frac{\tau(v_j - v_i)}{\tau(j-i)} \frac{\tau(v_i + v_j)}{\tau(i+j)}$$

A q-Nekrasov–Okounkov analogue for type $\widetilde{C}$

$$\delta_\lambda := (-1)^{D(\lambda)}$$

Theorem [W., 2022]

$$\sum_{\lambda \in \mathcal{DD}} \delta_\lambda T^{|\lambda|/2} \prod_{s\in\lambda} \frac{1-q^{h_s - 2\varepsilon_s} u^{-2\varepsilon_s}}{1-q^{h_s}} \prod_{s\in\Delta} \frac{1+uq^{h_s/2+1}}{1+u^{-1}q^{h_s/2-1}}$$

$$= \prod_{m\geq 1}(1-T^m)\left[\prod_{r\geq 1}\left[\frac{(1-uq^rT^m)^r(1-u^{-1}q^rT^m)^r}{(1-q^{r+1}T^m)^r(1-q^{r-1}T^m)^r}\right.\right.$$

$$\times \frac{(1-u^2q^rT^m)^{r-\lfloor r/2\rfloor-1}(1-u^{-2}q^{-r}T^m)^{r-\lfloor r/2\rfloor -1}}{(1-uq^{r+1}T^m)^r(1-u^{-1}q^{-r-1}T^m)^r}$$

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$u=q^z$, $q\to 1 \Rightarrow \widetilde{C}$-N–O type formula (Pétréolle, 2016)

Sketch of proof

1. Macdonald (1972): analogues of Weyl denominator formula for affine root systems

$$\sum_{w \in W} \det(w) e^{w(\rho)-\rho} = \prod_{\substack{a>0 \\ a \in R}} \left(1 - e^{-a}\right)$$

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   Special role of Δ !
3. Technical simplifications on the sum and the product side
4. Conclusion using a polynomiality argument with respect to the variable $u = q^t$