Genocchi numbers and hyperplane arrangements

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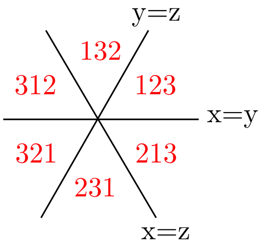
Joint work with Alex Lazar

The Braid Arrangement

The braid arrangement (or type A Coxeter arrangement) is the hyperplane arrangement in $\mathbb{R}^n$ defined by

$$\mathcal{A}_{n-1} = \{x_i - x_j = 0 : 1 \leq i < j \leq n\}.$$



Regions are open cones of form

$$R_\sigma := \{\mathbf{x} \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\}, \text{ where } \sigma \in \mathfrak{S}_n.$$

So $\mathcal{A}_{n-1}$ has $|\mathfrak{S}_n| = n!$ regions.

Some deformations of the braid arrangement

$$\mathcal{A}_{n-1} = \{x_i - x_j = 0 : 1 \leq i < j \leq n\} \subseteq \mathbb{R}^n.$$

- Linial arrangement: $x_i - x_j = 1$
- Shi arrangement: $x_i - x_j = 0, 1$
- Catalan arrangement: $x_i - x_j = -1, 0, 1$
- Semiorder arrangement: $x_i - x_j = -1, 1$

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Theorem (Postnikov-Stanley (2000))

 # regions of the Linial arrangement in $\mathbb{R}^n$ is equal to # alternating trees on $[n+1] := \{1, 2, \ldots, n+1\}$.

An alternating tree is a tree in which each node is either greater than all its neighbors or smaller than all its neighbors.

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- Linial arrangement: $x_i - x_j = 1$ — # alternating trees
- Shi arrangement: $x_i - x_j = 0, 1$ — $(n+1)^{n-1}$
- Catalan arrangement: $x_i - x_j = -1, 0, 1$ — $n! C_n$
- Semiorder arrangement: $x_i - x_j = -1, 1$ — # semiorders on $[n]$

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Hetyei's homogenized Linial arrangement

Hyperplane arrangement in $\mathbb{R}^{2n}$ defined by

$$\mathcal{H}_{2n-3} := \{x_i - x_j = y_i : 1 \leq i < j \leq n\}.$$

If we intersect $\mathcal{H}_{2n-3}$ with

- the subspace $y_1 = \cdots = y_n = 0$, we get the braid arrangement
- the subspace $y_1 = \cdots = y_n = 1$, we get the Linial arrangement

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Theorem (Hetyei (2017))

 # regions of $\mathcal{H}_{2n-1}$ is equal to the median Genocchi number h_n.

The Genocchi numbers

n	1	2	3	4	5	6
g_n	1	1	3	17	155	2073
h_n	2	8	56	608	9440	198272

$$\sum_{n\geq 1} g_n \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}$$

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Barsky-Dumont (1979):

$$\sum_{n\geq 1} g_n x^n = \sum_{n\geq 1} \frac{(n-1)!n!\,x^n}{\prod_{k=1}^{n}(1+k^2x)}$$

$$\sum_{n\geq 1} h_n x^n = \sum_{n\geq 1} \frac{n!(n+1)!x^n}{\prod_{k=1}^{n}(1+k(k+1)x)}$$

Combinatorial definition - Dumont 1974

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Genocchi numbers:

$$g_n = |\{\sigma \in \mathfrak{S}_{2n-2} : i \leq \sigma(i) \text{ if } i \text{ is odd}; \;\; i > \sigma(i) \text{ if } i \text{ is even}\}|.$$

These are called Dumont permutations.

$$g_3 = |\{(1,2)(3,4),\; (1,3,4,2),\; (1,4,2)(3)\}| = 3.$$

median Genocchi numbers:

$$h_n = |\{\sigma \in \mathfrak{S}_{2n+2} : i < \sigma(i) \text{ if } i \text{ is odd}; \;\; i > \sigma(i) \text{ if } i \text{ is even}\}|.$$

These are called Dumont derangements.

$$h_1 = |\{(1,2)(3,4),\; (1,3,4,2)\}| = 2.$$

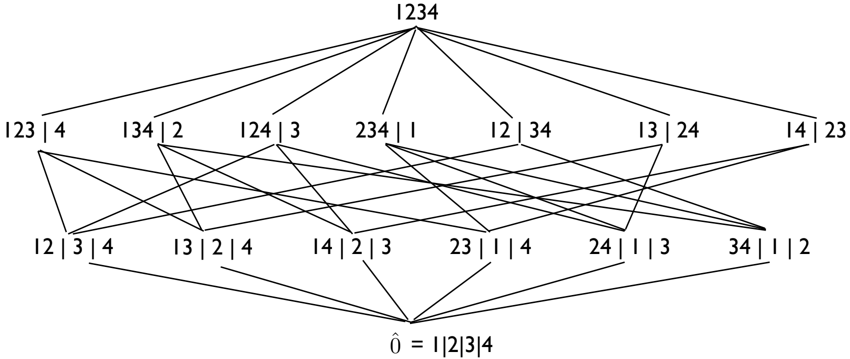
Zaslavsky's formula for the number of regions

Let $L(\mathcal{A})$ be the lattice of intersections of the hyperplane arrangement $\mathcal{A}$ ordered by reverse inclusion.

For the braid arrangement

$$\mathcal{A}_{n-1} = \{x_i - x_j = 0 : 1 \leq i < j \leq n\} \subseteq \mathbb{R}^n,$$

$L(\mathcal{A}_{n-1})$ is the partition lattice Π_n



Zaslavsky's formula for the number of regions

Let P be a finite ranked poset of length r with a minimum element $\hat{0}$. Define the characteristic polynomial of P to be

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{r - \mathrm{rk}(x)},$$

where $\mu_P(x, y)$ is the Möbius function and $\mathrm{rk}(x)$ is the rank of x.

Theorem (Zaslavsky (1975))

Let $\mathcal{A}$ be a hyperplane arrangement. The number of regions of $\mathcal{A}$ is equal to $|\chi_{L(\mathcal{A})}(-1)|$.

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For the braid arrangement $\mathcal{A}_{n-1}$,

$$\chi_{L(\mathcal{A}_{n-1})}(t) = \chi_{\Pi_n}(t) = \sum_{k=1}^{n} s(n, k) t^{k-1}$$

where $s(n, k)$ is the Stirling number of the first kind, which is equal to $(-1)^{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with exactly k cycles.

Zaslavsky's formula: $\#$regions of $\mathcal{A} = |\chi_{L(\mathcal{A})}(-1)|$

Hetyei's approach to proving that the number of regions of $\mathcal{H}_{2n-1}$ equals h_n:

- He uses the finite field method of Athanasiadis to obtain a recurrence for $\chi_{L(\mathcal{H}_{2n-1})}(t)$.
- When $t = -1$, the recurrence reduces to a known recurrence for the median Genocchi numbers.

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Our approach:

- We show that $t\chi_{L(\mathcal{H}_{2n-1})}(t)$ equals the chromatic polynomial $\mathrm{ch}_{\Gamma_n}(t)$ of a certain graph Γ_n.
- We show that the coefficients of $\mathrm{ch}_{\Gamma_n}(t)$ can be described in terms of a class of alternating forests.
- We construct a bijection from this class of alternating forests to a class of Dumont-like permutations.
- We construct a bijection from the Dumont-like permutations to a class of surjective staircases that is known to be enumerated by the median Genocchi number h_n.

Our approach also yields

Theorem (Lazar-W.)

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1})}(t)\, x^n = \sum_{n\geq 1} \frac{(t-1)_{n-1}(t-1)_n\, x^n}{\prod_{k=1}^{n}(1-k(t-k)x)},$$

where $(a)_n$ *denotes the falling factorial* $a(a-1)\cdots(a-n+1)$.

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Set $t=-1$. We get Barsky-Dumont generating function for h_n:

$$\sum_{n\geq 1} (-\chi_{L(\mathcal{H}_{2n-1})}(-1))x^n = \sum_{n\geq 1} \frac{n!(n+1)!\, x^n}{\prod_{k=1}^{n}(1+k(k+1)x)}.$$

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Set $t=0$. We get Barsky-Dumont generating function for g_n:

$$\sum_{n\geq 1} (-\chi_{L(\mathcal{H}_{2n-1})}(0))x^n = \sum_{n\geq 1} \frac{(n-1)!n!\, x^n}{\prod_{k=1}^{n}(1+k^2x)}.$$

Thus $\mu(L(\mathcal{H}_{2n-1})) = -g_n$

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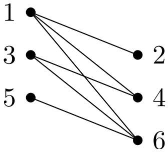
We also obtain type B analogs and Dowling arrangement generalizations of these type A results.

Chromatic polynomial $\mathrm{ch}_{\Gamma_n}(t)$

Let Γ_n be the bipartite graph on vertex set $\{1, 3, \ldots, 2n-1\} \sqcup \{2, 4, \ldots, 2n\}$ with an edge between $2i-1$ and $2j$ for all $i \leq j$.



Whitney (1932): For any graph G on vertex set $[n]$,

$$\mathrm{ch}_G(t) = t\chi_{L_G}(t),$$

where L_G is the bond lattice of G, that is, the induced subposet of the partition lattice Π_n consisting of partitions whose blocks induce connected subgraphs of G.

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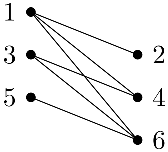

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Theorem (Lazar-W.)

$L(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice L_{Γ_n} of Γ_n. Consequently

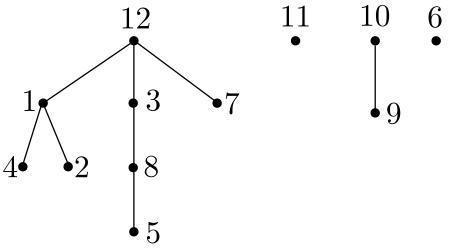
$$\chi_{L(\mathcal{H}_{2n-1})}(t) = \chi_{L_{\Gamma_n}}(t) = t^{-1}\mathrm{ch}_{\Gamma_n}(t).$$

Increasing-decreasing forests

The Rota-Whitney NBC theorem is used compute $\mathrm{ch}_{\Gamma_n}(t)$.

A rooted forest on finite node set $A \subset \mathbb{Z}^+$ is increasing-decreasing (ID) if the trees are rooted at their largest node and for each $a \in A$,

- if a is odd then a is less than all its descendants and all its children are even.
- if a is even then a is greater than all its descendants and all its children are odd.



Increasing-decreasing forests

With an appropriate ordering of the edges of Γ_n, the NBC sets are the ID-forests.

Theorem (Lazar-W)

Let $\mathcal{F}_{2n,k}$ be the set of ID forests on $[2n]$ with k trees. Then

$$\chi_{\mathcal{H}_{2n-1}}(t) = t^{-1}\mathrm{ch}_{\Gamma_n}(t) = \sum_{k=1}^{2n}(-1)^k|\mathcal{F}_{2n,k}|t^{k-1}.$$

Consequently,

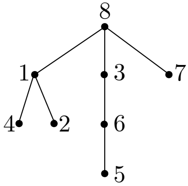
$$-\chi_{\mathcal{H}_{2n-1}}(0) = \#\ \textit{ID trees on } [2n]$$

$$\#\ \textit{regions of } \mathcal{H}_{2n-1} = -\chi_{\mathcal{H}_{2n-1}}(-1) = \#\ \textit{ID forests on } [2n]$$

A map from the ID forests on $[2n]$ to $\mathfrak{S}_{2n}$

Let T be an ID tree on node set A. Order the children of each even node of T in increasing order and the children of each odd node in decreasing order.

T =



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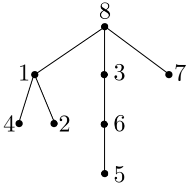

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Let $\alpha := \alpha_1, \cdots, \alpha_{|A|}$ be the word obtained by traversing T in postorder.

Now let $\psi(T)$ be the permutation whose cycle form is (α).

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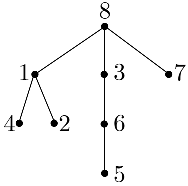

$$\psi(T) = (4, 2, 1, 5, 6, 3, 7, 8)$$

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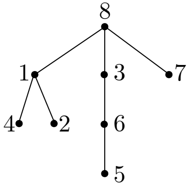

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Extend ψ to the set $\mathcal{F}_{2n}$ of ID forests on $[2n]$. We can show that $\psi : \mathcal{F}_{2n} \to \mathfrak{S}_{2n}$ is injective.

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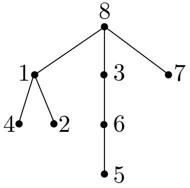

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Extend ψ to the set $\mathcal{F}_{2n}$ of ID forests on $[2n]$. We can show that $\psi : \mathcal{F}_{2n} \to \mathfrak{S}_{2n}$ is injective. What is the range?

The range of $\psi : \mathcal{F}_{2n} \to \mathfrak{S}_{2n}$: Dumont-like permutations

We say $\sigma \in \mathfrak{S}_{2n}$ is a D-permutation if $i \leq \sigma(i)$ whenever i is odd and $i \geq \sigma(i)$ whenever i is even.

Example: $\psi(T) = (4, 2, 1, 5, 6, 3, 7, 8)$ is a D-cycle.

$\mathcal{D}_{2n} = \{D\text{-permutations on } [2n]\}$, $\quad \mathcal{DC}_{2n} = \{D\text{-cycles on } [2n]\}$.

Note

$\mathcal{DC}_{2n} \subseteq \{\text{Dumont derange. on } [2n]\} \subseteq \{\text{Dumont perm. on } [2n]\} \subseteq \mathcal{D}_{2n}$.

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We show that the range of ψ is $\mathcal{D}_{2n}$. Hence

$$\psi : \mathcal{F}_{2n} \to \mathcal{D}_{2n}$$

is a bijection that takes a forest $\{T_1, \ldots, T_k\}$ in $\mathcal{F}_{2n}$ to a permutation in $\mathcal{D}_{2n}$ whose cycles are $\psi(T_1), \ldots, \psi(T_k)$.

The characteristic polynomial

Theorem (Lazar-W)

$$\chi_{L(\mathcal{H}_{2n-1})}(t) = \sum_{\sigma \in \mathcal{D}_{2n}} (-1)^{cyc(\sigma)} \ t^{cyc(\sigma)-1},$$

where $cyc(\sigma)$ is the number of cycles in σ.

Corollary

$$\mu(L(\mathcal{H}_{2n-1})) = |\mathcal{DC}_{2n}|.$$

$$\# \text{ regions } \mathcal{H}_{2n-1} = |\mathcal{D}_{2n}|.$$

D-permutations and Genocchi numbers

From the last slide

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Recall

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Recall

$$\mathcal{DC}_{2n} \subseteq \underbrace{\{\text{Dumont derange. on } [2n]\}}_{h_{n-1}} \subseteq \underbrace{\{\text{Dumont perm. on } [2n]\}}_{g_{n+1}} \subseteq \mathcal{D}_{2n}.$$

D-permutations and Genocchi numbers

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Corollary

$$\mu(L(\mathcal{H}_{2n-1})) = |\mathcal{DC}_{2n}| = g_n$$

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Recall

$$\underset{g_n}{\mathcal{DC}_{2n}} \subseteq \underset{h_{n-1}}{\{\text{Dumont derange. on } [2n]\}} \subseteq \underset{g_{n+1}}{\{\text{Dumont perm. on } [2n]\}} \subseteq \underset{h_n}{\mathcal{D}_{2n}}.$$

Theorem (Lazar-W)

$|\mathcal{D}_{2n}| = h_n$

We use the theory of surjective staircases to prove this.

Surjective staircases - Dumont (1992)

	1	2	3	4	5	6	7	8	9	10
10					X		X		X	X
8		X						X		
6						X				
4			X	X						
2	X									

one X in each column

at least one X in each row.

Some statistics:

- even maxima
  $em = 1$
- odd maxima
  $om = 3$
- double fixed points
  $dfix = 3$
- single fixed points
  $sfix = 1$

Generalized Dumont-Foata polynomial:

$$P_{2n}(x_1, x_2, x_3, x_4) = \sum_{F \in \mathcal{X}_{2n}} x_1^{om(F)+1} x_2^{sfix(F)+1} x_3^{em(F)} x_4^{dfix(F)}$$

Surjective staircase: $P_{2n}(\mathbf{x}) = \sum_{F\in\mathcal{X}_{2n}} x_1^{om(F)+1} x_2^{sfix(F)+1} x_3^{em(F)} x_4^{dfix(F)}$

Theorem (Randrianarivony-Zeng (1996))

$$\sum_{n\geq 1} P_{2n}(\mathbf{x}) z^n = \sum_{n\geq 1} \frac{(x_1)^{(n)}(x_3+x_4)^{(n)} z^n}{\prod_{k=1}^{n}(1+(x_1+k)(x_3+x_4-x_2+k)z)},$$

where $(x)^{(n)} = x(x+1)\ldots(x+n-1)$.

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where $(x)^{(n)} = x(x+1)\dots(x+n-1)$.

Using various bijections we obtain,

$$\sum_{\sigma \in \mathcal{D}_{2n}} t^{cyc(\sigma)} = P_{2n}(t, t+1, t+1, 0)$$

Surjective staircase: $P_{2n}(\mathbf{x}) = \sum_{F\in\mathcal{X}_{2n}} x_1^{om(F)+1} x_2^{sfix(F)+1} x_3^{em(F)} x_4^{dfix(F)}$

Theorem (Randrianarivony-Zeng (1996))

$$\sum_{n\geq 1} P_{2n}(\mathbf{x}) z^n = \sum_{n\geq 1} \frac{(x_1)^{(n)}(x_3+x_4)^{(n)} z^n}{\prod_{k=1}^{n}(1+(x_1+k)(x_3+x_4-x_2+k)z)},$$

where $(x)^{(n)} = x(x+1)\ldots(x+n-1)$.

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Recall $\chi_{L(\mathcal{H}_{2n-1})}(t) = \sum_{\sigma\in\mathcal{D}_{2n}} (-1)^{cyc(\sigma)} t^{cyc(\sigma)-1}$.

Theorem (Lazar-W)

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1})}(t)\, z^n = \sum_{n\geq 1} \frac{(t-1)_{n-1}(t-1)_n\, z^n}{\prod_{k=1}^{n}(1-k(t-k)z)}.$$

Another model for the (median) Genocchi numbers

Recall

$\sigma \in \mathfrak{S}_{2n}$ is a D-permutation if $i \leq \sigma(i)$ whenever i is odd and $i \geq \sigma(i)$ whenever i is even.

$\sigma \in \mathfrak{S}_{2n}$ is a a E-permutation if $i > \sigma(i)$ implies i is even and $\sigma(i)$ is odd.

$$\{(1,2)(3,4),\ (1,2,4)(3),\ (1,3,4)(2),\ (1,2)(3)(4),\ (1,4)(2)(3),\ (3,4)(1)(2),\ (1,2,3,4),\ (1)(2)(3)(4)\}$$

$$\mathcal{EC}_{2n} = \{E\text{-cycles on } [2n]\}, \qquad \mathcal{E}_{2n} = \{E\text{-permutations on } [2n]\}.$$

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Theorem (Lazar-W.)

$h_n = |\mathcal{D}_{2n}| = |\mathcal{E}_{2n}|$
$g_n = |\mathcal{DC}_{2n}|$

Conjecture (Lazar-W.)

$g_n = |\mathcal{EC}_{2n}|$

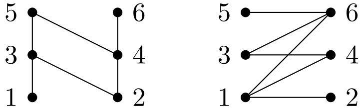
Conjecture proved by Lin-Yan (2021) and Pan-Zeng (2021).

Another model for the (median) Genocchi numbers

Recall Γ_n is the bipartite graph on vertex set $\{1, 3, \ldots, 2n-1\} \sqcup \{2, 4, \ldots, 2n\}$ with an edge between $2i-1$ and $2j$ for all $i \leq j$.

Observation: Γ_n is the incomparability graph of the poset P_n on $[2n]$ with order relation given by $x <_{P_n} y$ if:

- $x < y$ and $x \equiv y \mod 2$
- $x < y$, x is even, and y is odd.



P_3 $\qquad$ $\Gamma_3 = inc(P_3)$

Another model for the (median) Genocchi numbers

A permutation σ of the vertices of a poset P has a P-drop at x if $x >_P \sigma(x)$.

Chung-Graham (1995): For any finite poset P,

$$\mathrm{ch}_{\mathrm{inc}(P)}(t) = \sum_{k=0}^{|P|-1} d(P,k)\binom{k+t}{|P|},$$

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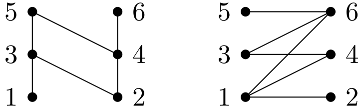
Set $t=-1$,

$$\mathrm{ch}_{\mathsf{inc}(P)}(-1) = \sum_{k=0}^{|P|-1} d(P,k)\binom{k-1}{|P|} = (-1)^{|P|} d(P,0).$$

Another model for the (median) Genocchi numbers



P_3 $\Gamma_3 = inc(P_3)$

Example: The cycle (532164) has P_3-drops at 5, 3, 6 only. Not 2

A permutation in $\sigma \in \mathfrak{S}_{2n}$ has no P_n-drops if for all $i \in [2n]$,
$i > \sigma(i)$ implies i is even and $\sigma(i)$ is odd, i.e. $\sigma \in \mathcal{E}_{2n}$

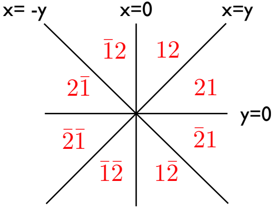
We have

$$h_n = \mathrm{ch}_{\Gamma_{2n}}(-1) = \mathrm{ch}_{inc(P_n)}(-1) = d(P_n, 0) = |\mathcal{E}_{2n}|$$

Type B

The type B braid arrangement in $\mathbb{R}^n$:

$$\mathcal{B}_n = \{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}.$$



$\#$ regions $= |B_n| = 2^n n!$

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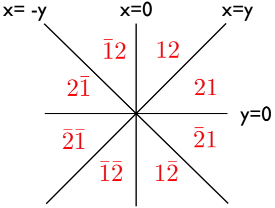

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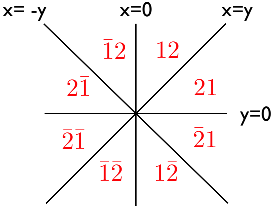

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$\#$ regions $= ?$ We can describe this with signed ID forests and signed D-permutations.

Type B

The type B analog of

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1})}(t)\, z^n = \sum_{n\geq 1} \frac{(t-1)_{n-1}(t-1)_n\, z^n}{\prod_{k=1}^{n}(1-k(t-k)z)}$$

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Theorem (Lazar-W)

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By setting $t=-1$,

$$\sum_{n\geq 1} r_n^B z^n = \sum_{n\geq 1} \frac{(2n)!\, z^n}{\prod_{k=1}^{n}(1+2k(2k+1)z)}.$$

By setting $t=0$,

$$\sum_{n\geq 1} \mu(L(\mathcal{H}^B_{2n-1}))\, z^n = \sum_{n\geq 1} \frac{(2n-1)!\, z^n}{\prod_{k=1}^{n}(1+(2k)^2 z)}.$$

Dowling arrangement

Let $\omega = e^{\frac{2\pi i}{m}}$. The Dowling arrangement $\mathcal{A}_n^m$ in $\mathbb{C}^n$:

$$\{x_i - \omega^l x_j = 0 : 1 \leq i < j \leq n,\ 0 \leq l < m\} \cup \{x_i = 0 : 1 \leq i \leq n\}.$$

- $\mathcal{A}_n^1$ is the complexified braid arrangement $\mathcal{A}_n$
- $\mathcal{A}_n^2$ is the complexified type B braid arrangement $\mathcal{B}_n$.

The intersection lattice $L(\mathcal{A}_n^m)$ is isomorphic to the classical Dowling lattice $Q_n(\mathbb{Z}_m)$.

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The homogenized Linial-Dowling arrangement $\mathcal{H}_{2n-1}^m$ in $\mathbb{C}^{2n}$:

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$\mathcal{H}_{2n-1}^2$ is a complexified version of $\mathcal{H}_{2n-1}^B$.

The intersection lattice $L(\mathcal{H}_{2n-1}^m)$

- We show that the intersection lattice is isomorphic to a subposet of the Dowling lattice $Q_n(\mathbb{Z}_m)$, analogous to the bond lattice of Γ_n.
- We describe the coefficients of the characteristic polynomial in terms of m-labeled ID-forests and m-labeled D-permutations.
- Then we use the correspondence between D-permutations and surjective staircases to obtain the following general formula.

Theorem (Lazar-W)

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1}^m)}(t)\, z^n = \sum_{n\geq 1} \frac{(t-1)_{n,m}(t-m)_{n-1,m}\, z^n}{\prod_{k=1}^n (1 - mk(t-mk)z)}.$$

where $(a)_{n,m} = a(a-m)(a-2m)\cdots(a-(n-1)m)$.

This reduces to the type A and type B generating function formulas when $m = 1, 2$.

m-analog of Genocchi numbers

$$g_n(m) = -\chi_{L(\mathcal{H}^m_{2n-1})}(0), \qquad h_n(m) = -\chi_{L(\mathcal{H}^m_{2n-1})}(-1)$$

n	$g_n(m)$	$h_n(m)$
0		1
1	1	2
2	m	$4(m+1)$
3	$m^2(m+2)$	$4(m+1)(m^2+4m+2)$
4	$m^3(3m^2+8m+6)$	$4(m+1)(3m^4+17m^3+32m^2+20m+4)$

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Theorem (Lazar-W)

$$g_n(m) = m^{2n-1} G_n(m^{-1}),$$

where $G_n(x)$ is the nth Gandhi polynomial.