Two different proofs of the Merca conjectures

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Introduction

At a conference in Brasov (Romania) organised by Alin Bostan and Killian Raschel, Mircea Merca posed a series of conjectures which caught the attention of the audience. Professor Christian Krattenthaler and myself were part of the audience. Later during that day Professor Krattenthaler said to me that he can reduce all the identities to identities which I could prove using my expertise. The conjectures which are now stated as theorems will be shown in the next slides.

Theorem 1

Theorem (conjectured in [7, Id. 5.1])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers m such that* $840m+361$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n}=\frac{(q,q^6,q^7;q^7)_\infty}{(q,q^4;q^5)_\infty}, \tag{1}$$

where

$$t(n)=\begin{cases}0, & \text{if } n\equiv 0,1,3,5,10,12,14,15 \pmod{16},\\ 1, & \text{otherwise.}\end{cases}$$

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i),$$

$$(a_1,a_2,\dots,a_m;q) := \prod_{j=1}^{m}(a_j;q)_\infty.$$

Theorem (conjectured in [7, Id. 5.2])

Let $(a_n)_{n\ge0}$ *be the sequence of non-negative integers* m *such that* $840m+529$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n}=\frac{(q,q^6,q^7;q^7)_\infty}{(q^2,q^3;q^5)_\infty}, \qquad (2)$$

where

$$t(n)=\begin{cases}0, & \text{if } n\equiv 0,2,3,6,9,12,13,15 \pmod{16},\\ 1, & \text{otherwise.}\end{cases}$$

Theorem 3

Theorem (conjectured in [7, Id. 5.3, corrected])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers m such that* $840m+121$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+4)/8\rfloor}q^{a_n}=\frac{(q^2,q^5,q^7;q^7)_\infty}{(q,q^4;q^5)_\infty}. \tag{3}$$

Theorem 4

Theorem (conjectured in [7, Id. 5.4])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers m such that* $840m+289$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n}=\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2,q^3;q^5)_\infty}, \qquad (4)$$

where

$$t(n)=\begin{cases}0, & \text{if } n\equiv 0,1,3,5,10,12,14,15 \pmod{16},\\ 1, & \text{otherwise.}\end{cases}$$

Theorem 5

Theorem (conjectured in [7, Id. 5.5, corrected])

Let $(a_n)_{n\geq 0}$ be the sequence of non-negative integers m such that $840m+1$ is a square. Then

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+4)/8\rfloor} q^{a_n} = \frac{(q^3,q^4,q^7;q^7)_\infty}{(q,q^4;q^5)_\infty}. \tag{5}$$

Theorem 6

Theorem (conjectured in [7, Id. 5.6])

Let $(a_n)_{n\ge0}$ be the sequence of non-negative integers m such that $840m+169$ is a square. Then

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n}=\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2,q^3;q^5)_\infty}, \tag{6}$$

where

$$t(n)=\begin{cases}0, & \text{if } n\equiv 0,1,2,4,11,13,14,15 \pmod{16},\\ 1, & \text{otherwise.}\end{cases}$$

Theorem 7

Theorem (conjectured in [7, Id. 6.1, corrected])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers m such that* $240m+1$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+2)/4\rfloor}q^{a_n}=\frac{(q,q^7,q^8;q^8)_\infty\,(q^6,q^{10};q^{16})_\infty}{(q,q^4;q^5)_\infty}. \tag{7}$$

Theorem 8

Theorem (conjectured in [7, Id. 6.2])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers* m *such that* $240m + 49$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor 5n/4 \rfloor} q^{a_n} = \frac{(q, q^7, q^8; q^8)_\infty \, (q^6, q^{10}; q^{16})_\infty}{(q^2, q^3; q^5)_\infty}. \tag{8}$$

Theorem 9

Theorem (conjectured in [7, Id. 6.5, corrected])

Let $(a_n)_{n\ge 0}$ *be the sequence of non-negative integers* m *such that* $240m+121$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+2)/4\rfloor}q^{a_n}=\frac{(q^3,q^5,q^8;q^8)_\infty\,(q^2,q^{14};q^{16})_\infty}{(q,q^4;q^5)_\infty}. \qquad (9)$$

Theorem 10

Theorem (conjectured in [7, Id. 6.6])

Let $(a_n)_{n\geq 0}$ *be the sequence of non-negative integers m such that* $240m + 169$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor 5n/4\rfloor} q^{a_n} = \frac{(q^3, q^5, q^8; q^8)_\infty \,(q^2, q^{14}; q^{16})_\infty}{(q^2, q^3; q^5)_\infty}. \qquad (10)$$

Theorem 11

Theorem (conjectured in [7, Id. 6.3, corrected])

Let $(a_n)_{n\ge 0}$ *be the sequence of non-negative integers* m *such that* $15m+1$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+2)/4\rfloor}q^{a_n}=\frac{(q^2,q^6,q^8;q^8)_\infty\,(q^4,q^{12};q^{16})_\infty}{(q,q^4;q^5)_\infty}. \qquad (11)$$

Theorem 12

Theorem (conjectured in [7, Id. 6.4, corrected])

Let $(a_n)_{n\ge 0}$ *be the sequence of non-negative integers m such that* $15m+4$ *is a square. Then*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor (n+2)/4\rfloor}q^{a_n}=\frac{(q^2,q^6,q^8;q^8)_\infty\,(q^4,q^{12};q^{16})_\infty}{(q^2,q^3;q^5)_\infty}. \qquad (12)$$

Theorem 13

Theorem (conjectured in [7, Id. 6.7])

We have

$$\sum_{n=-\infty}^{\infty} q^{n(5n+1)} = \frac{(q, q^9, q^{10}; q^{10})_\infty \, (q^8, q^{12}; q^{20})_\infty}{(q, q^4; q^5)_\infty}. \tag{13}$$

Theorem 14

Theorem (conjectured in [7, Id. 6.8])

We have

$$\sum_{n=0}^{\infty}\left(q^{n(n+1)}-q^{5n(n+1)+1}\right)=\frac{(q,q^9,q^{10};q^{10})_\infty\,(q^8,q^{12};q^{20})_\infty}{(q^2,q^3;q^5)_\infty}. \tag{14}$$

Theorem 15

Theorem (conjectured in [7, Id. 6.9])

We have

$$1+\sum_{n=1}^{\infty}\left(q^{n^2}+q^{5n^2}\right)=\frac{(q^2,q^8,q^{10};q^{10})_\infty\,(q^6,q^{14};q^{20})_\infty}{(q,q^4;q^5)_\infty}. \qquad (15)$$

Theorem 16

Theorem (conjectured in [7, Id. 6.10])

We have

$$\sum_{n=-\infty}^{\infty} q^{n(5n+2)} = \frac{(q^2, q^8, q^{10}; q^{10})_\infty \, (q^6, q^{14}; q^{20})_\infty}{(q^2, q^3; q^5)_\infty}. \qquad (16)$$

Theorem 17

Theorem (conjectured in [7, Id. 6.11])

We have

$$\sum_{n=0}^{\infty}\left(q^{n(n+1)}+q^{5n(n+1)+1}\right)=\frac{(q^3,q^7,q^{10};q^{10})_\infty\,(q^4,q^{16};q^{20})_\infty}{(q,q^4;q^5)_\infty}. \tag{17}$$

Theorem (conjectured in [7, Id. 6.12])

We have

$$\sum_{n=-\infty}^{\infty} q^{n(5n+3)} = \frac{(q^3, q^7, q^{10}; q^{10})_\infty \, (q^4, q^{16}; q^{20})_\infty}{(q^2, q^3; q^5)_\infty}. \tag{18}$$

Theorem 19

Theorem (conjectured in [7, Id. 6.13])

We have

$$\sum_{n=-\infty}^{\infty} q^{n(5n+4)} = \frac{(q^4, q^6, q^{10}; q^{10})_\infty \, (q^2, q^{18}; q^{20})_\infty}{(q, q^4; q^5)_\infty}. \qquad (19)$$

Theorem 20

Theorem (conjectured in [7, Id. 6.14])

We have

$$\sum_{n=1}^{\infty}\left(q^{n^2-1}-q^{5n^2-1}\right)=\frac{(q^4,q^6,q^{10};q^{10})_\infty\,(q^2,q^{18};q^{20})_\infty}{(q^2,q^3;q^5)_\infty}. \quad (20)$$

Taking into consideration that there are 20 different identities that appear to be similar in nature, one naturally is led to suggest an algorithmic proof method, in contrast to looking at each identity individually and trying to find an optimal proof based on its particular form . Algorithmic proofs are part of RISC (Research Institute for Symbolic Computation) expertise.

$$E_g = E_g(q;N) := q^{NB_2(g/N)/2}(q^g, q^{N-g}; q^N)_\infty, \qquad (21)$$

where $B_2(x) = x^2 - x + \frac{1}{6}$. Each theorem presented above is transformed into an identity of the form

$$\sum_{j=1}^{r} c_j \prod_g E_g^{a_g^{(j)}} = 0. \qquad (22)$$

where $a_g^{(j)}$ satisfies the following conditions (see next slide).

$$\sum_g a_g^{(j)} \equiv 0 \pmod{12} \quad \text{and} \quad \sum_g g^2 a_g^{(j)} \equiv 0 \pmod{y(N)}, \tag{23}$$

where $y(N) = 2N$ if N is even, and $y(N) = N$ if N is odd. When (23) are satisfied a Theorem by Yifan Yang gives that

$$\sum_{j=1}^{r} c_j \prod_g E_g^{a_g^{(j)}}$$

is a modular function for the group $\Gamma_1(N)$.

Theorem

Let f be a holomorphic modular function, and let $A_N \subset \mathbb{Q} \cup \{\infty\}$ be a complete set of cusps for $\Gamma_1(N)$, with $\infty \in A_N$. Assume that $\sum_{a \in A_N} \operatorname{Ord}(f; a, N) > 0$. Then f is the zero function.

Theorem (YifanYang)

$$\operatorname{Ord}(E_g; c, N) = \frac{1}{2} \gcd(D_c, N) B_2(\{N_c g / \gcd(D_c, N)\}), \tag{24}$$

where D_c is the denominator of c and N_c is the numerator of c, while $\{\alpha\}$ denotes the fractional part of the rational number α. In particular the order of $\prod_g E_g^{a_g^{(j)}}$ at c then is

$$\sum_g a_g^{(j)} \operatorname{Ord}(E_g; c, N).$$

Corollary

Let $f := \sum_{j=1}^{r} c_j \prod_g E_g^{a_g^{(j)}}$. *Then*

$$\sum_{a \in A} \operatorname{Ord}(f; a, N)$$

$$> \operatorname{Ord}(f; \infty, N) + \sum_{a \in A_N \setminus \{\infty\}} \min_{j \in \{1,\dots,r\}} \sum_g a_g^{(j)} \operatorname{Ord}(E_g; a, N).$$

In particular

$$\operatorname{Ord}(f; \infty, N) + \sum_{a \in A_N \setminus \{\infty\}} \min_{j \in \{1,\dots,r\}} \sum_g a_g^{(j)} \operatorname{Ord}(E_g; a, N) > 0$$

implies that $f = 0$.

Note that $\operatorname{Ord}(f; \infty, N) = m$, if $f = c_m q^m + c_{m+1} q^{m+1} + \dots$ and $c_m \neq 0$.

Let $f := \sum_{j=1}^{r} c_j \prod_g E_g^{a_g^{(j)}}$, and define the bound for f, $Bound(f)$ by

$$Bound(f) := \sum_{a \in A_N \setminus \{\infty\}} \min_{j \in \{1,\dots,r\}} \sum_g a_g^{(j)} \operatorname{Ord}(E_g; a, N) > 0,$$

By the above corollary if $\operatorname{Ord}(f; \infty, N) > -Bound(f)$, then $f = 0$. To prove that $f = 0$, we need to compute the set A_N, luckily the computer algebra Magma does this for us. Next, we need to compute $Bound(f)$, which depends on A_N, finally we check by computer that $f = 0 + 0q + 0q^2 + \cdots + 0q^{-B(f)} + \ldots$.

Proof of Theorem 1

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n} =$$

$$\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+19)^2-361)} + \sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+61)^2-361)}$$

$$-\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+79)^2-361)} + \sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+89)^2-361)}$$

$$-\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+121)^2-361)} + \sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+131)^2-361)}$$

$$-\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+149)^2-361)} - \sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{840}((210k+191)^2-361)}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{19k}{2}} + \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{61k}{2} + 4} - \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{79k}{2} + 7} \\
&+ \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{89k}{2} + 9} - \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{121k}{2} + 17} + \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{131k}{2} + 20} \\
&- \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{149k}{2} + 26} - \sum_{k=0}^{\infty} (-1)^k q^{\frac{105k^2}{2} + \frac{191k}{2} + 43}. \qquad (25)
\end{aligned}
$$

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n} = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{105k^2}{2}+\frac{19k}{2}} + \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{105k^2}{2}+\frac{61k}{2}+4}$$
$$- \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{105k^2}{2}+\frac{79k}{2}+7} + \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{105k^2}{2}+\frac{89k}{2}+9}.$$

Jacobi triple product identity

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n} = (q^{105},q^{62},q^{43};q^{105})_\infty + q^4\,(q^{105},q^{83},q^{22};q^{105})_\infty$$
$$- q^7\,(q^{105},q^{92},q^{13};q^{105})_\infty + q^9\,(q^{105},q^{97},q^{8};q^{105})_\infty. \quad (26)$$

Theorem 1 reduces to this

$$0 = (q^{105}, q^{62}, q^{43}; q^{105})_\infty + q^4\,(q^{105}, q^{83}, q^{22}; q^{105})_\infty$$
$$-q^7\,(q^{105}, q^{92}, q^{13}; q^{105})_\infty + q^9\,(q^{105}, q^{97}, q^{8}; q^{105})_\infty - \frac{(q, q^6, q^7; q^7)_\infty}{(q, q^4; q^5)_\infty}.$$

Always divide everything by one of the terms

$$0 = 1 + q^4 \frac{(q^{105}, q^{83}, q^{22}; q^{105})_\infty}{(q^{105}, q^{62}, q^{43}; q^{105})_\infty} - q^7 \frac{(q^{105}, q^{92}, q^{13}; q^{105})_\infty}{(q^{105}, q^{62}, q^{43}; q^{105})_\infty}$$
$$+ q^9 \frac{(q^{105}, q^{97}, q^{8}; q^{105})_\infty}{(q^{105}, q^{62}, q^{43}; q^{105})_\infty} - \frac{(q, q^6, q^7; q^7)_\infty}{(q, q^4; q^5)_\infty (q^{105}, q^{62}, q^{43}; q^{105})_\infty}.$$

to make sure that each term is a modular function.

$$0 = 1 + \frac{E_{22}}{E_{43}} - \frac{E_{13}}{E_{43}} + \frac{E_8}{E_{43}} - \frac{E_7 E_8 E_{13} E_{15} E_{20} E_{22} E_{27} E_{28} E_{35} E_{42} E_{48} E_{50}}{E_4 E_9 E_{11} E_{16} E_{19} E_{24} E_{26} E_{31} E_{39} E_{44} E_{46} E_{51}}. \tag{27}$$

In order to rewrite the last term we used that

$$(q^7; q^7) = (q^7, q^{14}, q^{21}, \dots, q^{105}; q^{105}),$$

and similar "blow-ups" for other terms. We set

$$f := 1 + \frac{E_{22}}{E_{43}} - \frac{E_{13}}{E_{43}} + \frac{E_8}{E_{43}} - \frac{E_7 E_8 E_{13} E_{15} E_{20} E_{22} E_{27} E_{28} E_{35} E_{42} E_{48} E_{50}}{E_4 E_9 E_{11} E_{16} E_{19} E_{24} E_{26} E_{31} E_{39} E_{44} E_{46} E_{51}}. \tag{28}$$

and pretend that $f \neq 0$.

The cusps of $\Gamma_1(105)$, A_{105}

[oo,0,1/13,1/12,2/23,1/11,3/32,2/21,1/10,3/29,5/48,2/19,
3/28,4/37,1/9,5/44,4/35,3/26,8/69,5/43,2/17,3/25,4/33,1/8,
6/47,5/39,9/70,4/31,11/84,13/99,5/38,7/53,2/15,13/96,8/59,
3/22,7/51,1/7,5/34,4/27,29/195,18/121,3/20,48/319,79/525,
5/33,16/105,7/45,18/115,8/51,4/25,17/105,1/6,6/35,
11/63,18/103,7/40,8/45,23/129,5/28,7/39,9/50,11/60,9/49,
12/65,5/27,19/102,30/161,13/69,17/90,4/21,1/5,109/525,
27/130,19/91,23/110,22/105,47/222,18/85,33/155,3/14,
14/65,68/315,13/60,41/189,64/295,29/133,19/87,26/119,23/105
9/41,20/91,11/50,11/49,9/40,71/315,23/102,30/133,
17/75,27/119,8/35,13/56,44/189,7/30,13/55,5/21,6/25,
9/35,11/42,59/225,37/140,23/87,53/200,13/49,4/15,62/231,
51/190,47/175,32/119,368/1365,17/63,13/48,29/105,31/112,
46/165,41/147,59/210,69/245,2/7,13/45,17/63,71/245,7/24,
92/315,45/154,43/147,31/105,34/115,29/98,52/175,
25/84,94/315,19/63,32/105,13/42,24/77,11/35,16/45,113/315,
48/133,38/105,23/63,11/30,31/84,13/35,37/98,8/21,67/175,

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523/1365,29/75,64/165,41/105,124/315,2/5,43/105,41/100,
26/63,31/75,44/105,103/245,47/105,16/35,7/15,8/15,19/35,
39/70,137/245,47/84,17/30,4/7,97/168,41/70,37/63,13/21,
152/245,87/140,22/35,19/30,24/35,46/63,11/15,23/30,27/35].
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Last step

Based on this $Bound(f) = -148$. It is routine to verify that $f = 0 + 0q + \cdots + 0q^{148} + \cdots$. This implies that f must be the zero function.

All 20 theorems are proven in this way, this has been done and the paper was sent back to professor Krattenthaler. Normally at this point the paper would have been finished (from my point of view) but Professor Krattenthaler did not send any email for a while after sending him this. The reason comes in the next slides.

Second proof idea by professor Krattenthaler

$$\theta(\alpha; q) := (\alpha, q/\alpha; q)_\infty.$$

Weierstraß' addition formula:

$$\begin{aligned}\theta(xy;q)\,\theta(x/y;q)\,\theta(uv;q)\,\theta(u/v;q)-&\\ \theta(xv;q)\,\theta(x/v;q)\,\theta(uy;q)\,\theta(u/y;q)&\\ =\frac{u}{y}\,\theta(yv;q)\,\theta(y/v;q)\,\theta(xu;q)\,\theta(x/u;q).&\quad(29)\end{aligned}$$

Two specialisations of the Weierstrass formula

$$(u^3/q^N, q^{4N}/u^3, q^{3N}; q^{3N})_\infty + \frac{q^N}{u}(u^3/q^{2N}, q^{5N}/u^3, q^{3N}; q^{3N})_\infty = \frac{(u^2/q^N, q^{2N}/u^2, q^N; q^N)_\infty}{(u, q^N/u; q^N)_\infty}. \quad (30)$$

$$(u^3/q^{2N}, q^{5N}/u^3, q^{3N}; q^{3N})_\infty - \frac{q^{3N}}{u^2}(u^3/q^{4N}, q^{7N}/u^3, q^{3N}; q^{3N})_\infty = \frac{(u^2/q^{2N}, q^{3N}/u^2, q^N; q^N)_\infty}{(u/q^N, q^{2N}/u; q^N)_\infty}. \quad (31)$$

Proof of Theorem 1 Our point of departure is (26). By (30) with $N = 35$ and $u = q^{26}$, respectively with $N = 35$ and $u = q^{19}$, we get

$$\sum_{n=0}^{\infty}(-1)^{t(n)}q^{a_n} = \frac{(q^{17},q^{18},q^{35};q^{35})_\infty}{(q^{26},q^{9};q^{35})_\infty} + q^4\frac{(q^{3},q^{32},q^{35};q^{35})_\infty}{(q^{19},q^{16};q^{35})_\infty}.$$

If we now replace q by q^{35} and choose $u = q^{10}$, $v = q^3$, $x = q^{14}$, and $y = q^6$ in (29), we obtain

$$\begin{aligned}\theta(q^{17};q^{35})\,\theta(q^{11};q^{35})\,\theta(q^{16};q^{35})\,\theta(q^{4};q^{35})&\\ +q^4\theta(q^{9};q^{35})\,\theta(q^{3};q^{35})\,\theta(q^{24};q^{35})\,\theta(q^{4};q^{35})&\\ =\theta(q^{20};q^{35})\,\theta(q^{8};q^{35})\,\theta(q^{13};q^{35})\,\theta(q^{7};q^{35})&,\end{aligned}$$

and thus the above right-hand side becomes

$$\frac{\theta(q^{20};q^{35})\,\theta(q^{8};q^{35})\,\theta(q^{13};q^{35})\,\theta(q^{7};q^{35})\,(q^{35};q^{35})_\infty}{\theta(q^{16};q^{35})\,\theta(q^{9};q^{35})\,\theta(q^{11};q^{35})\,\theta(q^{4};q^{35})},$$

which is equivalent to the right-hand side of (1).

All other theorems are proven in a similar fashion. At this point we felt that our expertise was not really needed in this problem. However professor Krattenthaler suggested that we could still be of help for the last substitution, there four parameters needed to be found in order to match the formula and although professor Krattenthaler could find almost all of them by hand, two of them gave problems, and he asked me whether I could automatize this. It took me some days to make a program based on exhaustive search which could find all substitutions automatically, by this time professor Krattenthaler was done also with the last two examples. The only advantage of our program was that we could find substitutions of the form q^r with r a positive integer while some of the substitutions found by professor Krattenthaler had r a half integer.

Thank you!