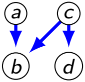
The world of poset inequalities

Greta Panova (University of Southern California)

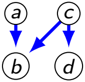
AEC Konferenz, Wien, Juli 2022

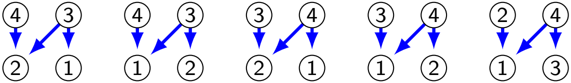
Linear extensions of posets

Poset $P =$ 

Linear extensions of posets

Linear extensions $\mathcal{E}(P)$

Poset $P =$ 



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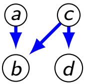
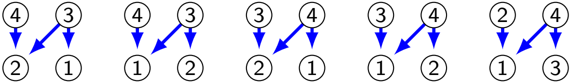
Poset $P =$ 



The **number of linear extensions** of P: $e(P) = |\mathcal{E}(P)|$

Linear extensions of posets

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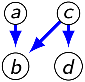
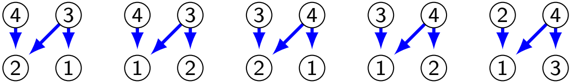
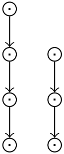
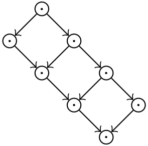
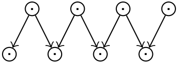
Poset $P =$ 



The **number of linear extensions** of P: $e(P) = |\mathcal{E}(P)|$

Chain $\mathcal{C}_n$	Antichain $\mathcal{A}_n$	$\mathcal{C}_m + \mathcal{C}_k$	"Catalan poset" $2 \times n$	Zigzag poset
				
$e(P) = 1$	$e(P) = n!$	$e(P) = \binom{m+k}{k}$	$e(P) = C_n = \frac{1}{n+1}\binom{2n}{n}$	$e(P) = E_{2n}$

Linear extensions of posets

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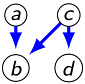
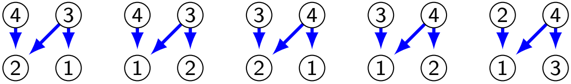
Poset $P =$ 



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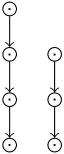
Antichain $\mathcal{A}_n$

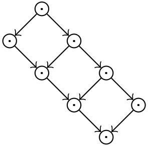
$e(P) = n!$

$\mathcal{C}_m + \mathcal{C}_k$



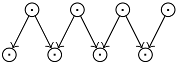
$e(P) = \binom{m+k}{k}$

"Catalan poset" $2 \times n$



$e(P) = C_n = \frac{1}{n+1}\binom{2n}{n}$

Zigzag poset



$e(P) = E_{2n}$

[Brightwell-Winkler]: Counting linear extensions is #P-complete.
[Dittmer-Pak]: Counting linear extensions of height 2 posets is #P-complete.

Inequalities:

Sorting probability of $x, y \in P$:

$$\Pr[x \prec y] = \frac{|\{L \in \mathcal{E}(P), \text{ s.t. } L(x) < L(y)\}|}{e(P)}$$

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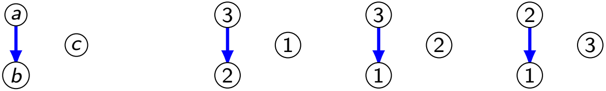

Inequalities: the $\frac{1}{3}$–$\frac{2}{3}$ conjecture

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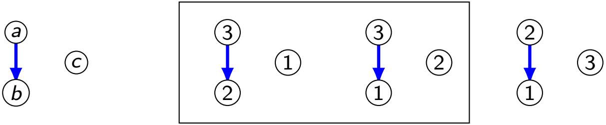

$$\Pr[c \prec a] = \tfrac{2}{3}$$

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For every finite poset that is not a total order, there exist x, y*:*

$$\frac{1}{3} \leq \Pr[x \prec y] \leq \frac{2}{3}.$$

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[Brightwell-Felsner-Trotter'95]: "This problem remains one of the most intriguing problems in the combinatorial theory of posets."

[Kahn-Saks' 84]: For every finite poset, which is not a chain, there exist x, y, s.t.

$$\frac{3}{11} \leq \Pr[x \prec y] \leq \frac{8}{11}$$

[Brightwell-Felsner-Trotter '95]:

$$\frac{5-\sqrt{5}}{10} \leq \Pr[x \prec y] \leq \frac{5+\sqrt{5}}{10}.$$

[Linial'84]: conjecture holds for width 2 posets, improved by [Sah'18]. Other special cases: [Brightwell'89], [Zaguia'12,'19], [Trotter, Gehrlein, Fishburn], [Pouzet]. Improvement by [Saks'85]. [Olson-Sagan'18] for (skew) Young diagrams.

Sorting probability for posets: asymptotics

Sorting probability of P:

$$\delta(P) = \min_{x,y \in P} |\Pr[x \prec y] - \Pr[y \prec x]|$$

Conjecture (Kahn-Saks'84)

For every finite poset P on n elements

$$\delta(P) \to 0 \qquad \text{as } width(P) \to \infty.$$

($width(P)$ = size of max antichain)
[Komlós '90] Proved for posets with $\Omega(\frac{1}{\log \log n})$ minimal elements.

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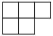
Theorem (Chan-Pak-Panova'21+)

Let $\lambda \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \epsilon n$. Let P_λ be the poset of the Young diagram of λ. Then there is a constants $C > 0$ depending on d, ϵ, s.t.

$$\delta(P_\lambda) < \frac{C}{\sqrt{n}}.$$

$\lambda = (3, 2) =$ 

$e(P_\lambda) = \#SYT(\lambda) = f^\lambda$

1	2	3
4	5	

1	2	4
3	5	

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3	4	

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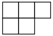
$\lambda = (3,2) =$ 

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1 2 3 / 4 5	1 2 4 / 3 5	1 2 5 / 3 4
1 2 5 / 3 4	1 3 5 / 2 4	

Theorem (Chan-Pak-Panova'21+)

Let $\lambda = (n,n)$, so P_λ is the "Catalan poset". Then

$$\delta(P_\lambda) \leq \frac{C}{n^{\frac{5}{4}}}.$$

Refined inequalities: log-concavity

Theorem (Stanley '81, Rivest conjecture)

Fix an element $x \in P$, set $N(k) := \#\{L \in \mathcal{P} : L(x) = k\}$. The

$$N(k)^2 \geq N(k-1)N(k+1).$$

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Fix elements $x, y \in P$*, let* $F(k) := \#\{L \in \mathcal{E}(P) : L(y) - L(x) = k\}$*. Then*

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Proofs via mixed-volumes in in the order polytope, Aleksandrov-Fenchel inequality.
[Chan-Pak-P'21]: Proof for width 2 posets via lattice paths injections.

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[Chan-Pak-P'21]: Proof for width 2 posets via lattice paths injections.

Conjecture (Cross-product inequality, Brightwell-Felsner-Trotter'95)

Fix $x, y, z \in P$. Let $F(k, \ell) := \#\{L \in \mathcal{E}(P) : L(y) - L(x) = k, L(z) - L(y) = \ell\}$. Then

$$F(k, \ell)F(k+1, \ell+1) \leq F(k, \ell+1)F(k+1, \ell).$$

[Chan-Pak-Panova, Trans. AMS'22]: Combinatorial proofs for width 2 posets via lattice path injections.

Log-concavity: some motivation

Fields Medals 2022

The Fields Medal is awarded to recognize outstanding mathematical achievement for existing work and for the promise of future achievement.
The medals and cash prizes are funded by a trust established by J.C.Fields at the University of Toronto, which has been replenished periodically, but is still significantly underfunded. In 2022, the prize funds from the University of Toronto were supplemented by generous support from the Heidelberg Laureate Forum Foundation/Klaus Tschira Stiftung.

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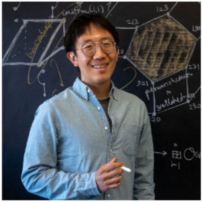

Photo credit: Lance Murphey

June Huh

For bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron–Rota–Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture.

citation | video | write-up | CV/publications | interview | laudatio

proceedings | Plus magazine! article

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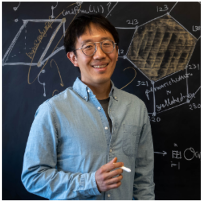

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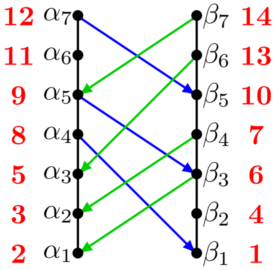
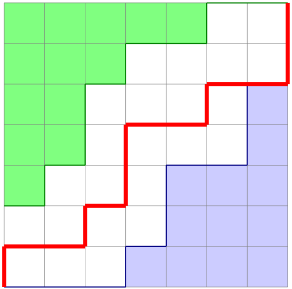
Theorem (Mason's conjecture, proven by Adiprasito-Huh-Katz, 2015)

Let (X, I) be a matroid, and $f(k) = \{A : A \in I, |A| = k\}$, then

$$f(k)^2 \geq f(k-1)f(k+1).$$

Combinatorial proofs:
Linear extension ↔ lattice paths





Lattice path inequalities

Lemma (CPP)

Consider lattice paths in a simply connected region R with monotone boundaries. Let A, A', B, B' be on the same grid line, C, D on another grid line to the right with $|AA'| = |B'B|$, A', B' between A, B, and such that if $AB||CD$ then $|AB'| \geq |CD|$. We then have:

$$Paths(A' \to C) \times Paths(B' \to D) \geq Paths(A \to C) \times Paths(B \to D).$$

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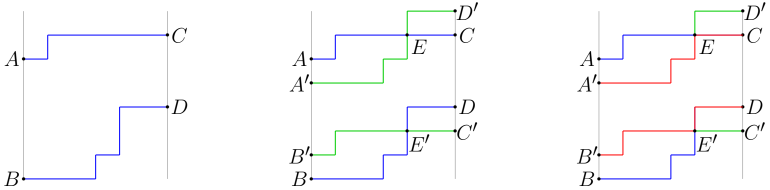
Proofs via injective maps:



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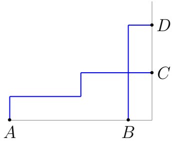

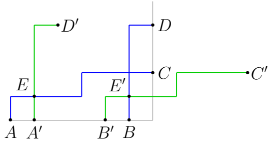

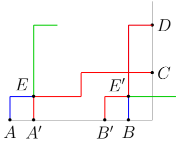

Stanley's inequality

Theorem (q–Stanley inequality, Chan-Pak-Panova'21+)

Let $P = (X, \prec)$ *be a finite poset of width two, let* $x \in X$*, and let* $X = \{\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b\}$ *with* $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_a$ *and* $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_b$ *be its two chains.*

$$N_q(k) := \sum_{L \in \mathcal{E}(P) \, : \, L(x)=k} q^{L(\alpha_1)+\cdots+L(\alpha_a)} .$$

Then:

$$N_q(k)^2 \geqslant N_q(k-1)\, N_q(k+1) \qquad \textit{for all} \quad k > 1,$$

where the inequality between polynomials in q *is coefficient-wise.*

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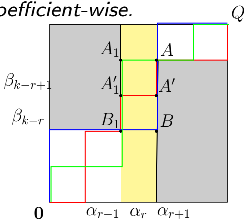
where the inequality between polynomials in q is coefficient-wise.

Proof:

let $x = \alpha_r$ and $A = (r-1, k-r+1)$, $A' = B' = (r-1, k-r)$, $B = (r-1, k-1-r)$, $A_1 = (r, k-r+1)$, $A_1' = B_1' = (r, k-r)$, $B_1 = (r, k-r-1)$.

$$N_q(k) = q^{\binom{a+1}{2}+k-r} \sum_{\gamma \in Reg(P): A', A_1' \in \gamma} q^{area(\gamma)}$$



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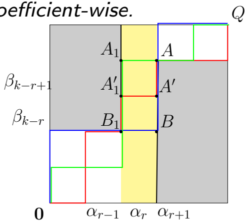

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$$N_q(k) = q^{\binom{a+1}{2}+k-r} \sum_{\gamma \in Reg(P): A', A_1' \in \gamma} q^{area(\gamma)}$$

$$N_q(k)^2 - N_q(k-1)N_q(k+1)$$
$$= (0 \to A')_q^2 (A_1' \to Q)_q^2 - (0 \to A)_q (0 \to B)_q (A_1 \to Q)_q (B_1 \to Q)_q$$

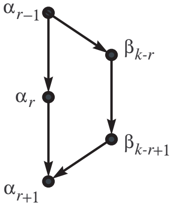
Equality in Stanley's inequality

Let $x = \alpha_r$. We say that x satisfies a k-**pentagon property** if

$$\alpha_{r-1} \prec \beta_{k-r} \prec \beta_{k-r+1} \prec \alpha_{r+1}$$

and

$$\alpha_r \,||\, \beta_{k-r} \ , \ \ \alpha_r \,||\, \beta_{k-r+1} \ ,$$



Theorem (Chan-Pak-Panova'21 [1])

Let $P = (X, \prec)$ be a finite poset of width two. Fix $x \in X$, and let $N(k)$, $N_q(k)$ be defined as earlier. Suppose that $N(k) > 0$. Then the following are equivalent:

1. $N(k)^2 = N(k-1)\,N(k+1)$,
2. $N(k) = N(k+1) = N(k-1)$,
3. $N_q(k)^2 = N_q(k-1)\,N_q(k+1)$,
4. $N_q(k) = q^{\epsilon} N_q(k-1) = q^{-\epsilon} N_q(k+1)$, *where* $\epsilon = \pm 1$
5. *element x satisfies k-pentagon property.*

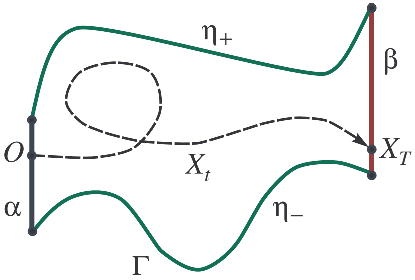
[1] Equivalence of 1. and 2. proven in general by Shenfeld-van Handel, analysis of Aleksandrov-Fenchel inequality.

Random Walks

Theorem (Chan-Pak-Panova'21)

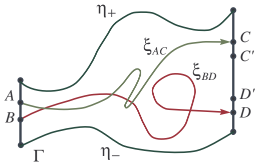
Let $\Gamma \subset \mathbb{Z}^2$ be a simply connected region with boundary $\partial\Gamma = \alpha \cup \eta_+ \cup \eta_- \cup \beta$, s.t. $\alpha \subset \{x = 0\}, \beta \subset \{x = m\}$ –vertical intervals, and η_+, η_- are two x-monotone lattice paths.
Let $\{X_t\}$ be the nearest neighbor lattice random walk, s.t. $X_0 = O \in \alpha$, and is absorbed whenever X_t tries to exit the region Γ. Denote by T the first time t such that $X_t \in \beta$, and let $p(k)$ be the exit probability that $X_T = (m, k)$. Then $\{p(k)\}$ is log-concave:

$$p(k)^2 \geq p(k+1)\, p(k-1) \qquad \text{for all } k \in \mathbb{Z}, \text{ such that } (m, k \pm 1) \in \beta.$$

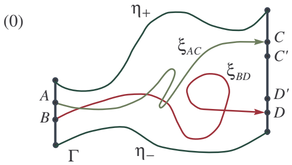


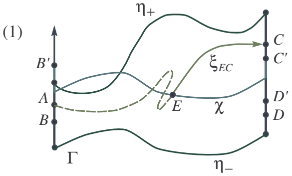
Random Walk injections

(0)



Random Walk injections





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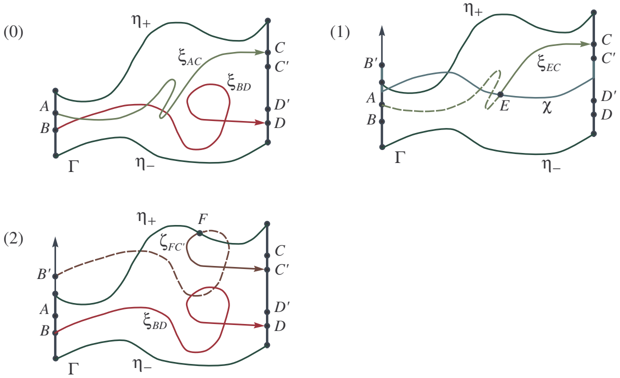

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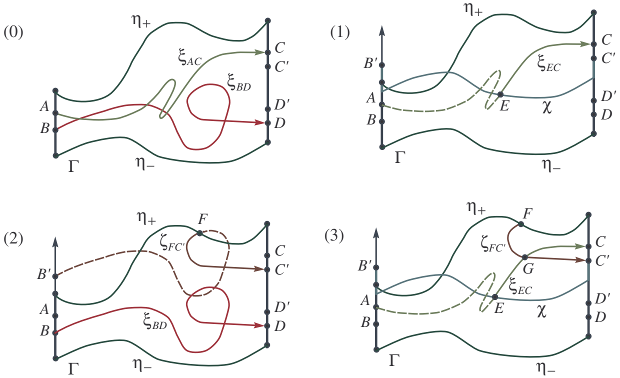

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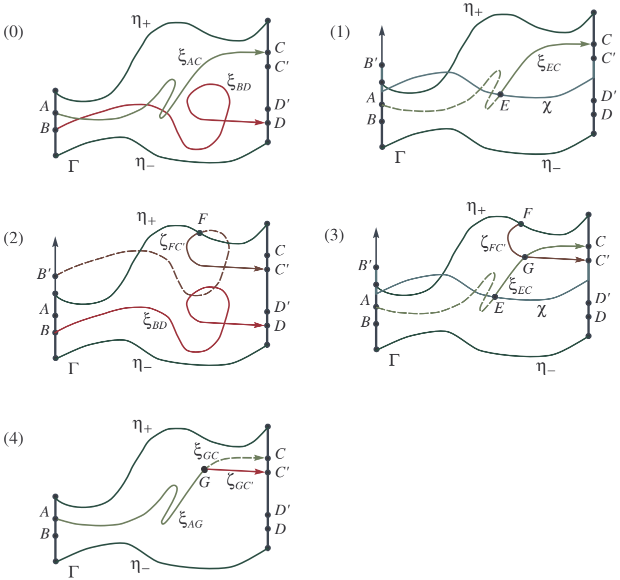

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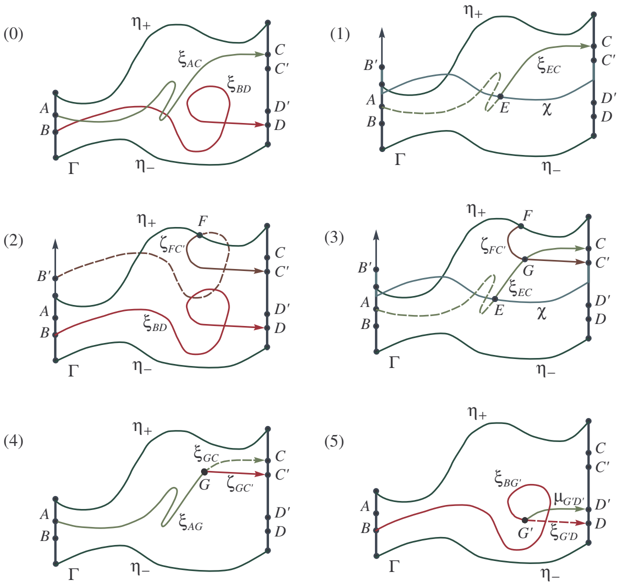


Generalizations: weights

Transition probabilities:

$$\Pr\big[(i,j) \to (i \pm 1, j)\big] \;=\; \pi_\pm(i,j), \qquad \Pr\big[(i,j) \to (i, j \pm 1)\big] \;=\; \omega_\pm(i,j),$$

$$\pi_+(i,j) \,+\, \pi_-(i,j) \,+\, \omega_+(i,j) \,+\, \omega_-(i,j) \,=1,$$

y-invariant:

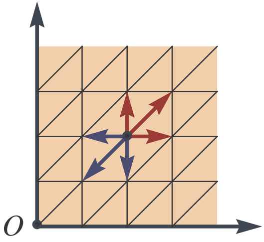
$$\pi_\pm(i,j) \,=\pi_\pm(i,j'), \quad \omega_\pm(i,j) \,=\, \omega_\pm(i,j') \quad \text{for all} \quad i,j \;\text{ and }\; j'.$$

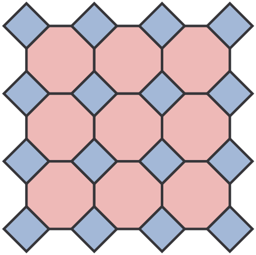
Theorem (Chan-Pak-Panova'21)

Let $\Gamma \subset \mathbb{Z}^2$ be the lattice region as before. Let $\{X_t\}$ be the lattice random walk in Γ starting at $X_0 = O \in \alpha$ with transition probabilities as above, absorbed if $X_t \notin \Gamma$. Let $T = \min\{t : X_t \in \beta\}$, and let $p(k) = \Pr[X_T = (m,k)]$. Then $\{p(k)\}$ is log-concave:

$$p(k)^2 \,\geq\, p(k+1)\,p(k-1) \qquad \textit{for all} \;\; k \in \mathbb{Z}, \;\; \textit{such that} \quad (m, k \pm 1) \in \beta.$$

Other lattices and regions





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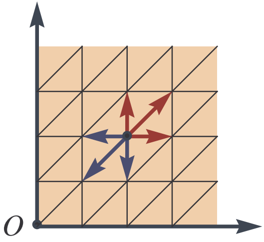

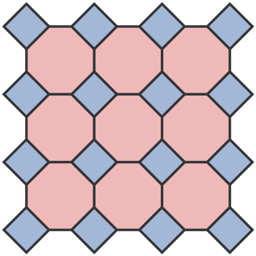

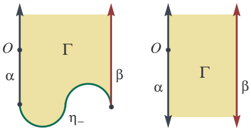

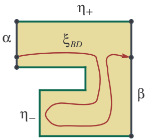

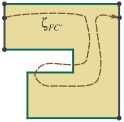

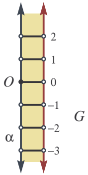

Order polynomial of P

$$\Omega(P,t) := \#\{g : P \to [1,\dots,t], g(x) \le g(y) \text{ if } x \prec_P y\}$$

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poset P	**Linear extensions** $e(P)$	**Order polynomial** $\Omega(P,t)$
chain on $[n]$: $\mathcal{C}_n$	1	$\binom{t+n-1}{n}$
antichain on $[n]$: $\mathcal{A}_n$	$n!$	t^n
$\mathcal{C}_m + \mathcal{C}_k$	$\binom{m+k}{k}$	$\binom{t+m-1}{m}\binom{t+k-1}{k}$
Catalan poset, $2 \times m$	$C_m = \frac{1}{m+1}\binom{2m}{m}$	$\frac{(t-1)t^2\cdots(t+m-1)^2(t+m)}{(m+1)!m!}$
$\mathcal{A}_{n-1} \oplus \mathcal{A}_1$	$(n-1)!$	$1^{n-1} + \cdots + t^{n-1}$
P_λ, $\lambda = a \times b$	$f^\lambda = \frac{n!}{\prod_{u\in[\lambda]} h_u}$	$s_{(a^b)}(1^t) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{t}\frac{i+j+k-1}{i+j+k-2}$

$\Omega(P, t)$ vs $e(P)$

Theorem (Chan-Pak-P'22+)

For every poset P on n elements we have

$$\Omega(P, t) \geq \frac{e(P)}{n!} t^n$$

with equality if and only if P is an antichain.

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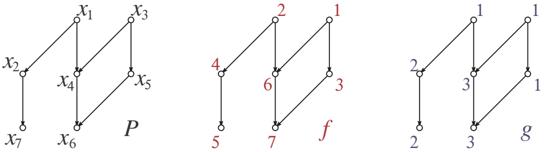
Proof:

Injection $\Psi : \mathcal{E}(P) \times [t]^n \to \Omega(P,t) \times S_n$

$$(f, \{\{2,3,7\},\{4,6\},\{1,5\}\}) \to (g, \sigma = 3274615)$$



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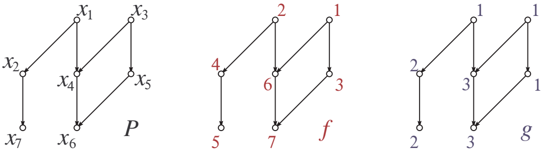

Corollary: $n!\Omega(P,t) - e(P)t^n \in \#\mathrm{P}$ ("combinatorial interpretation").

Björner-Wachs inequality

Upper order ideal: $B(x) := \{y \in P : y \succcurlyeq x\}$, $b(x) := |B(x)|$.

Theorem (Björner and Wachs)

Let P be a poset with n elements. Then

$$e(P) \;\geq\; n! \cdot \prod_{x \in X} \frac{1}{b(x)} \,.$$

with equality iff P is a top rooted forest.

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Theorem (Chan-Pak-P'22+)

Let P be a poset on n elements, and r be the number of its minimal elements. Then for every $t \in \mathbb{N}$ we have

$$\Omega(P,t) \geq t^r (t+1)^{n-r} \prod_{x \in P} \frac{1}{b(x)}$$

Proof: 1) If x– unique minimal, then

$$\frac{\Omega(P,t)}{t+1} \geq \frac{\Omega(P \setminus x, t)}{b(x)}$$

2) If x, y – minimal, then FKG (correlation) inequality gives:

$$\frac{\Omega(P,t)}{\Omega(P \setminus x, t)} \geq \frac{\Omega(P \setminus y, t)}{\Omega(P \setminus \{x,y\}, t)} \geq \cdots \geq \frac{t+1}{b(x)}$$

Log-concavity, revisited

$\Omega(P, t; x, a) := \#\{$ order preserving $g : P \to [t]$, s.t. $g(x) = a\}$

Conjecture (Graham, Chan-Pak-P'22+)

Let $x \in P$ be fixed. Then for every integer a we have

$$\Omega(P, t; x, a)^2 \geq \Omega(P, t; x, a+1)\Omega(P, t; x, a-1).$$

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Theorem (Chan-Pak-P'22+)

Let P be a poset on n elements. then for every integer $t \geq 2$ we have

$$\Omega(P, t)^2 \geq \left(1 + \frac{1}{(1+t)^{n+1}}\right)\Omega(P, t+1)\Omega(P, t-1)$$

Corollary: Conjecture holds for t sufficiently large.

Monotonicity

Conjecture (Kahn-Saks)

Let P be a poset on n elements. Then for every $t \in \mathbb{N}$ we have

$$\frac{\Omega(P,t)}{t^n} \geq \frac{\Omega(P,t+1)}{(t+1)^n}.$$

Conjecture[P]: $\Omega(P,t)/t^n$ is a weakly decreasing function for $t \in [1,+\infty)$.

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Theorem (Chan-Pak-P)

Suppose that P is such for which the conjecture holds. Let M be the set of minimal elements of P, $r := |M|$. Then

$$\Omega(P,t) \geq t^r \prod_{x \in P \setminus M} F_{b(x)}(t),$$

where $F_m(t) := \frac{1}{t^m} \sum_{k=1}^{t} k^m$.

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Proposition (Chan-Pak-P'22+)

For every $k, t \in \mathbb{N}$ then

$$\frac{1}{t^n}\Omega(P,t) \geq \frac{1}{(kt)^n}\Omega(P,kt)$$

and in fact $k^n\Omega(P,t) - \Omega(P,kt) \in \#\mathrm{P}$.

Proof: injection.

q-analogues

$$\Omega_q(P,t) := \sum_{g:P\to[t]} q^{|g|-n}$$

(order preserving maps g, $|g| := \sum_{x\in P} g(x)$)

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Theorem (Reiner'22+)

The following inequality between q-power series holds coefficient-wise:

$$\sum_{g:P\to\mathbb{N}} q^{|g|} \geq_q \prod_{x\in P} \frac{1}{1-q^{b(x)}}.$$

Vielen Dank für Ihre Aufmerksamkeit!