Perfect Matchings in Cubic Graphs

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This talk is dedicated to the memory of

GERT SABIDUSSI
1929 - 2022

Cycles, 2-factors, perfect matchings

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where, for a graph G, $PerMat(G)$ $\#$ of perfect matchings of G.

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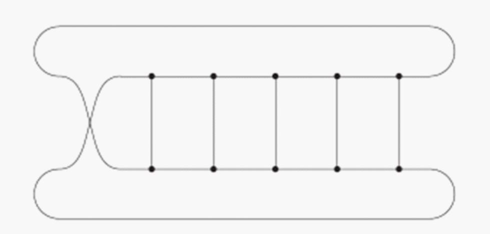

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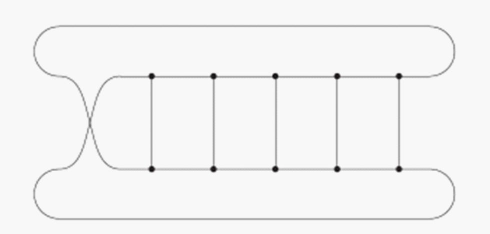
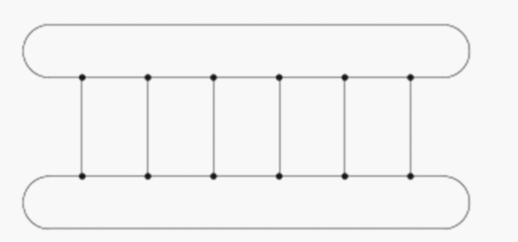


Circular ladder on 12 vertices $C_6 \times K_2$



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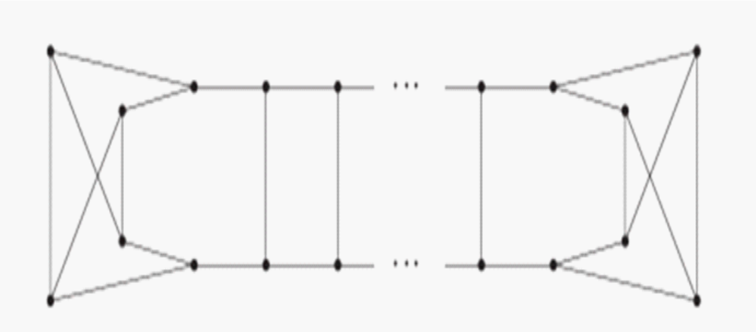
Extrem(n) a ladder with a $K_{3,3}$ included at both ends.



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(Bregman–Minc inequality 1973) *Let* $A = (a_{ij})$ *be a binary square matrix, row sum* $r_i = \sum_{1 \le j \le n} a_{ij}$ *for* $i = 1, ..., n$. *Then*

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$$PerMat(G) \le \prod_{i=1}^{n} 6^{1/3} = 6^{n/3} \text{ for cubic bipartite graphs}$$

A one-to-one correspondence between a binary square matrices and a simple balanced bipartite graphs. Each term of permanent $a_{1\sigma(1)} \ldots a_{n\sigma(n)} = 1$ corresponds to a perfect matching, and vice versa.

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A formula for PerMat(G)

Theorem

For a cubic graph G of order $2n$,

$$PerMat(G) = \mathrm{E}(X), \text{ the expected value of } X$$

where X is a random variable defined on the set of all 2-colorings c on the edges of G, each coloring equally likely, and $X(c) = (-3)^{m_c}$, where m_c is the number of vertices of G incident in c with three edges of the same color.

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Not feasible for practical calculation but might turn out to have a theoretical value.

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2-dimensional real vector space

$$B = \mathbb{R}|0\rangle \oplus \mathbb{R}|1\rangle = \{a|0\rangle + b|1\rangle : a, b \in \mathbb{R}\}$$

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and linear map β is given by

$$\beta \colon B \otimes B \otimes B \to \mathbb{R},$$
$$|s_1\rangle \otimes |s_2\rangle \otimes |s_3\rangle \mapsto \begin{cases} 1 & \text{there are exactly two 0 among } s_1, s_2, s_3, \\ 0 & \text{otherwise.} \end{cases}$$

where $s_1, s_2, s_3 \in \{0, 1\}$, basis of $B \otimes B \otimes B$.

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$$\alpha = |x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle$$

and the evaluation applies in this case as well.

Closing remarks

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Conjecture

Let G be a simple 3-connected cubic graph on $2n$ vertices, where $n \geq 3$. Then

$$PerMat(G) \leq f_n + 2f_{n-1} + 2,$$

with equality attained if and only if G is isomorphic to the circular ladder graph or the Möbius ladder graph

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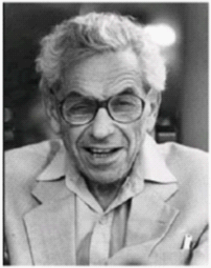
Unlike the maximal graphs, the graphs that minimize the number of perfect matchings tend to be complicated

Erdos - Dongryul Kim



1913 - 1996

1997 -