Reconstruction of polytopes and Kalai's conjecture on reconstruction of spheres

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Algorithmic Enumerative Combinatorics
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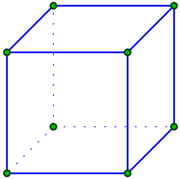
Outline

- Convex polytopes
- Kalai's conjecture
- Subword complexes
- Main result

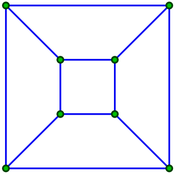
Convex polytopes

(Convex) polytope P:
convex hull of finitely many points in Euclidian space.

The graph $G(P)$:
the graph consisting of the vertices and edges of P.



P



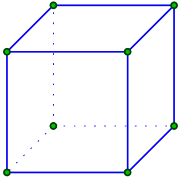
$G(P)$

Simple polytope P:
number of edges incident to each vertex equals the dimension of P.

Reconstruction of polytopes

Theorem (Blind–Mani, 1987)

If P is a simple polytope, then the graph $G(P)$ determines the entire combinatorial structure of P.



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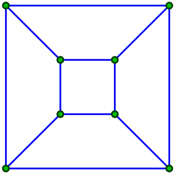

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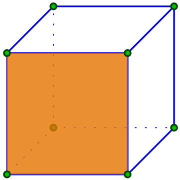

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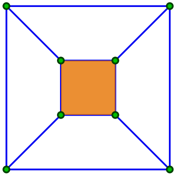

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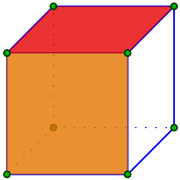

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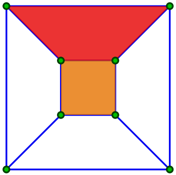

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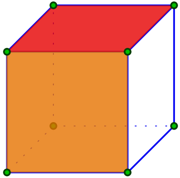

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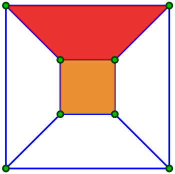

$G(P)$

Kalai, 1988: A simple constructive proof.

Reconstruction of polytopes

Theorem (Blind–Mani, 1987)

If P is a simple polytope, then the graph $G(P)$ determines the entire combinatorial structure of P.

Graph reconstruction holds for arbitrary polytopes (not just simple) in dimension 3.

Theorem (Steinitz, 1916)

A graph is the graph of a (unique) 3-polytope if and only if it is planar and 3-connected.

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General graph reconstruction does not hold in higher dimensions.

Example

Let Δ_m be a m-dimensional simplex. The following are two non isomorphic 6-dimensional polytopes with the same graph (complete graph on 7 vertices)

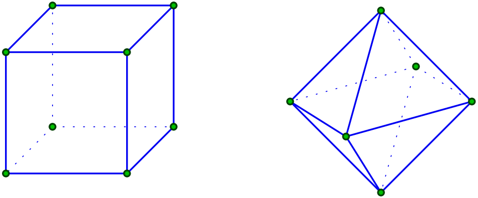
$$(\Delta_2 \times \Delta_4)^* \ncong (\Delta_3 \times \Delta_3)^*$$

Duality of polytopes

Every nonempty d-polytope P in $\mathbb{R}^d$ admits a dual polytope in $\mathbb{R}^d$:

$$P^* = \{y \in \mathbb{R}^d : x^T y \leq 1 \text{ for all } x \in P\}$$

where P is assumed to contain the origin in its interior.



P $\qquad$ P^*

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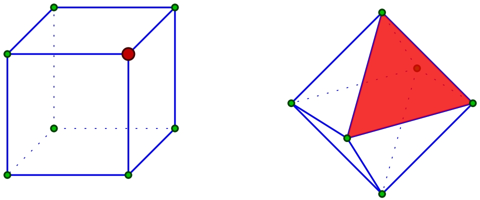

P P^*

Under this duality:

$$\begin{array}{ccc} P & \longleftrightarrow & P^* \\ \text{vertices} & \longleftrightarrow & \text{facets (higher dim faces)} \end{array}$$

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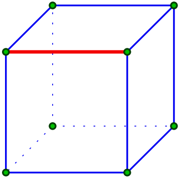

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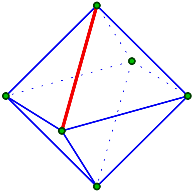

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P	$\longleftrightarrow$	P^*
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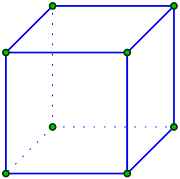

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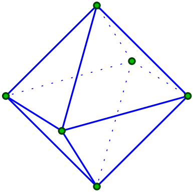

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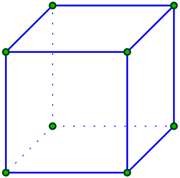
$\ldots$	$\longleftrightarrow$	$\ldots$

Simple vs simplicial

Simplicial polytope P:
all faces are simplices.

The facet-ridge graph $G_{FR}(P)$:
the graph whose vertices are facets of P
two facets are connected by an edge if they intersect in a ridge.

$$\begin{array}{ccc} P \text{ is simple} & \longleftrightarrow & P^* \text{ is simplicial} \\ G(P) & = & G_{FR}(P^*) \end{array}$$



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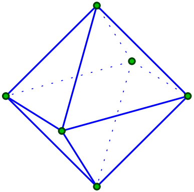

P^*

Reconstruction of polytopes and spheres

Theorem (Blind–Mani, 1987)

Simplicial polytopes are completely determined by their facet-ridge graphs.

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Conjecture (Blind–Mani, 1987; Kalai, 2009)

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Simplicial spheres are completely determined by their facet-ridge graphs.

A simplicial sphere is a simplicial complex which is homeomorphic to a sphere.

Most spheres are not polytopal

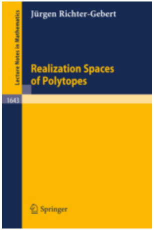
For $d \geq 3$, most d-spheres are not polytopal.

- Goodman–Pollack, 1986
- Kalai, 1988
- Pfeifle–Ziegler, 2004

Deciding polytopality of spheres is a difficult problem

Mnëv's Universality theorem: Realization spaces of polytopes can take arbitrary (semi-algebraic) shapes and thus can exhibit all kinds of pathologies.

The realizability problem for 4-polytopes is NP-hard.



Goal

Our initial goal was:
Look for a counterexample to Kalai's Conjecture among a special family of simplicial spheres which are conjectured to be polytopal.
(kill two conjectures at once)

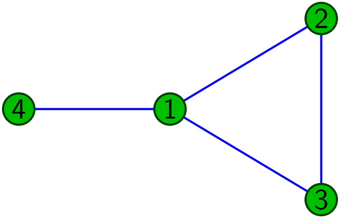
Instead:
We proved the conjecture for this family.
(spherical subword complexes)

Rest of the talk:
Introduce subword complexes, state our main result, give proof sketch.

Subword complexes preliminaries

Simplical Complex Δ: A collection of subsets of a ground set E, s.t. $\sigma \in \Delta$ and $\tau \in \sigma$ implies $\tau \in \Delta$

faces: subsets in Δ
vertices: singleton sets
facets: maximal sets
ridges: facets missing a single element



$\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}\}$

Subword complexes preliminaries

Symmetric group $\mathbb{S}_{n+1}$:
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length of w: smallest r such that $w = s_{i_1} \ldots s_{i_r}$
longest element: permutation $[n+1, \;\ldots\; , 1]$
reduced expression for w: expression for w of minimal length

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In this talk: **finite Coxeter groups**
(very similar to the symmetric group)

Subword complexes

W finite Coxeter group with generating set S

$Q = (q_1, \dots, q_m)$ a word in S

$\pi \in W$

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$\pi \in W$

Definition (Knutson–Miller, 2004)

The subword complex $\Delta(Q, \pi)$ is the simplicial complex whose

faces $\longleftrightarrow$ subwords P of Q such that $Q \setminus P$ contains a reduced expression of π

Knutson–Miller. Gröbner geometry of Schubert polynomials. Ann. Math., 161(3), '05
Knutson–Miller. Subword complexes in Coxeter groups. Adv. Math., 184(1), '04

Subword complexes - Example 1

In type A_2:

$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

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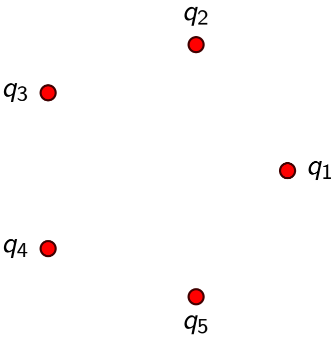

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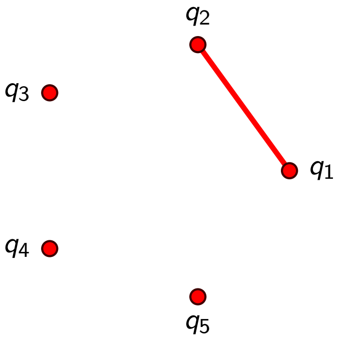

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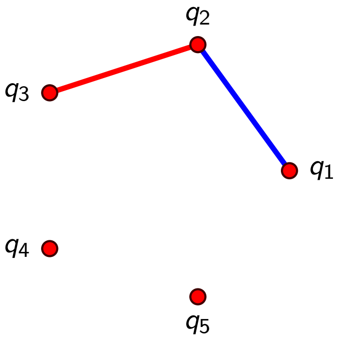

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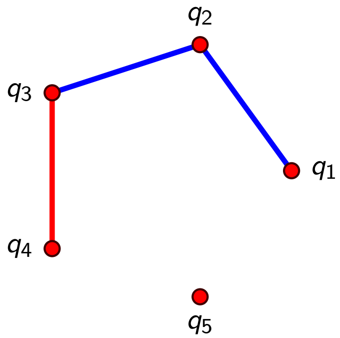

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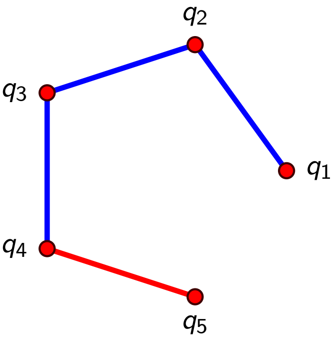

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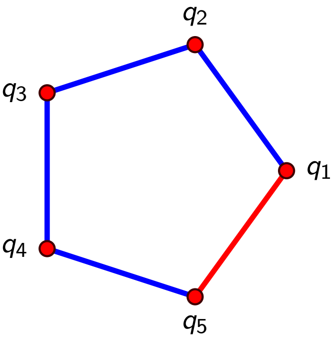

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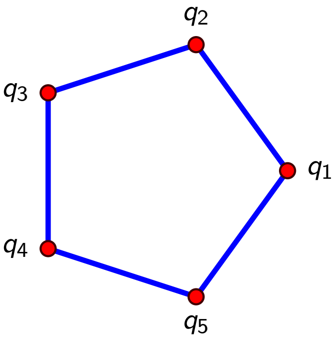

Subword complexes - Example 2

In type A_3:

$W = \mathbb{S}_4$, $S = \{s_1, s_2, s_3\} = \{(1\ 2), (2\ 3), (3\ 4)\}$

$$Q = \begin{array}{c} (\ s_1, s_2, s_1, s_2, s_1, s_3\ ) \\ q_1, q_2, q_3, q_4, q_5, q_6 \end{array} \text{ and } \pi = [3\ 2\ 1] = s_1 s_2 s_1 = s_2 s_1 s_2$$

Subword complexes - Example 2

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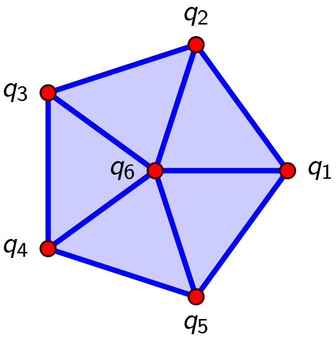

Subword complexes

Conjecture (Knutson–Miller, Ceballos–Labbé-Stump, ...)

Spherical subword complexes are polytopal.

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Spherical subword complexes are polytopal.

Special cases include:

- Cyclic polytopes
- Duals of associahedra
- Cluster complexes of cluster algebras of finite type
- Duals of pointed-pseudotriangulation polytopes
- Simplicial multi-associahedra (conjectured)

Woo, Pilaud–Pocchiola, Serrano–Stump, Stump, C.-Labbé–Stump, Rote–Santos–Streinu, Jonsson, ...

Our main theorem

Theorem (Ceballos–D.)

Spherical subword complexes of finite type are completely determined by their facet-ridge graph. In other words, they satisfy Kalai's Conjecture.

Our current proof is not constructive.
It is based on the topological tools developed by Blind and Mani.

Proof Sketch

For P, Q simplical spheres with a facet-ridge isomorphism f, define $g : P \to Q$ by

$$g(\sigma) = \bigcap \{f(F) | \text{F facet of } P, \sigma \subset F\}.$$

A simplicial complex is **strongly vertex decomposable** if for each vertex, the deletion and link of that vertex is **vertex decomposable**.

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A simplicial complex is **strongly vertex decomposable** if for each vertex, the deletion and link of that vertex is **vertex decomposable**.

A simplicial complex is **vertex decomposable** if it is pure, and for a vertex, the deletion and link of that vertex is vertex decomposable.

Proof Sketch

Lemma

If σ a $(d-2)$-face of P, either $g(\sigma)$ is a $(d-2)$-face of Q or $\tilde{H}_{d-2}(Q \setminus f(\sigma))$ is nontrivial.

Lemma

If P, Q are both strongly vertex decomposable, $\tilde{H}_{d-2}(Q \setminus f(\sigma))$ is trivial.

Proof Sketch

Theorem (Knutson-Miller, 2004)

Subword complexes are vertex decomposable spheres or balls.

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Theorem (Ceballos-D.)

Spherical subword complexes of finite type are strongly vertex decomposable.

Summary

- Simplicial spheres are conjectured to be completely determined by their facet-ridge graph.
- Simplicial polytopes are completely determined by their facet-ridge graph.
- Spherical subword complexes are conjectured to be polytopes.
- Spherical subword complexes of finite type are completely determined by their facet-ridge graph.