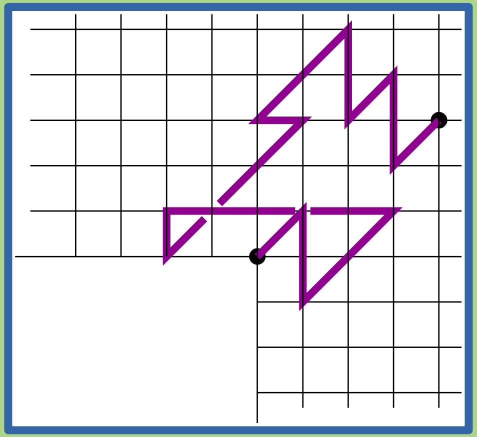
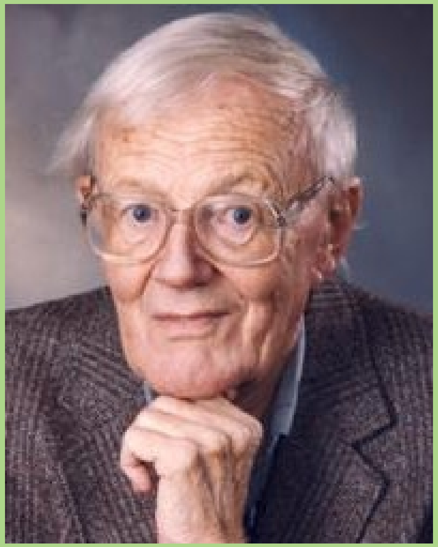
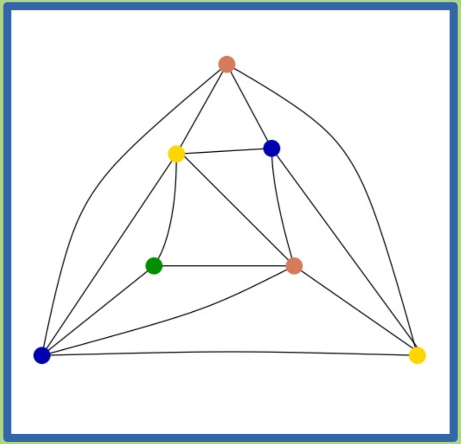
Tutte's invariants







Keywords: enumerative combinatorics, functional equations, algebraic series

Mireille Bousquet-Mélou
CNRS, Université de Bordeaux, France

In this talk

I. An equation

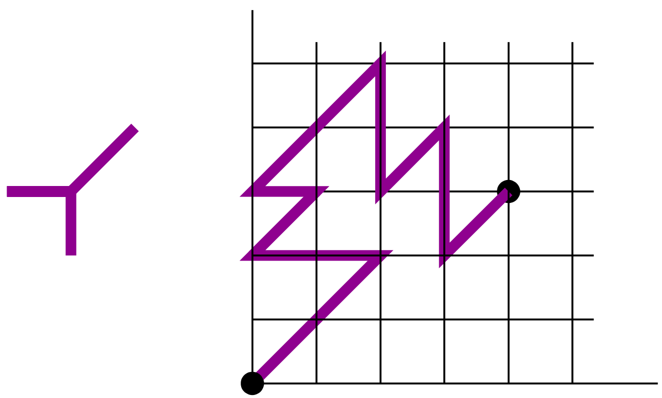
II. More equations with two catalytic variables

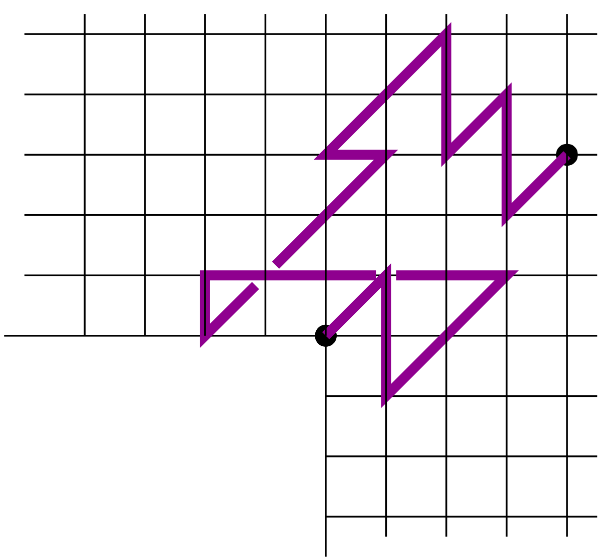
III. Catalytic variables: $0<1<2< \dots$

IV. Tutte's invariants

V. Quadrant walks: the whole picture

VI. Three-quadrant walks: a partial picture





I. An equation

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

- Defines a unique formal power series in t

$$Q(x,y) = 1 + txy + t\frac{1-1}{y} + t\frac{1-1}{x} + \mathcal{O}(t^2)$$

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + \boxed{t\frac{Q(x,y) - Q(0,y)}{x}}$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y

$$Q(x,y) = \sum_{n \geq 0} \left(\sum_{0 \leq i,j \leq n} q_{i,j}(n) x^i y^j \right) t^n$$

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

$$\left(1 - t\left(xy + \frac{1}{x} + \frac{1}{y}\right)\right) Q(x,y) = 1 - \frac{t}{y} Q(x,0) - \frac{t}{x} Q(0,y)$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

kernel

$$\left(1 - t\left(xy + \frac{1}{x} + \frac{1}{y}\right)\right) Q(x,y) = 1 - \frac{t}{y} Q(x,0) - \frac{t}{x} Q(0,y)$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

kernel

$$(1 - t(xy + \bar{x} + \bar{y}))Q(x,y) = 1 - t\bar{y}Q(x,0) - t\bar{x}Q(0,y)$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

Notation: $\bar{x} = 1/x, \quad \bar{y} = 1/y.$

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

kernel

$$(1 - t\,(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

Notation: $\bar{x} = 1/x, \quad \bar{y} = 1/y.$

An equation

Series $Q(t;x,y) \equiv Q(x,y)$:

kernel

$$(1 - t(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- Defines a unique formal power series in t
- Involves two divided differences w.r.t. x and y
- The coefficients are polynomials in x and y
- The variables x and y are said to be catalytic [Zeilberger 00]

  Tautological equation at $x=0$ and $y=0$

Notation: $\bar{x} = 1/x, \quad \bar{y} = 1/y.$

An equation for walks in a quadrant

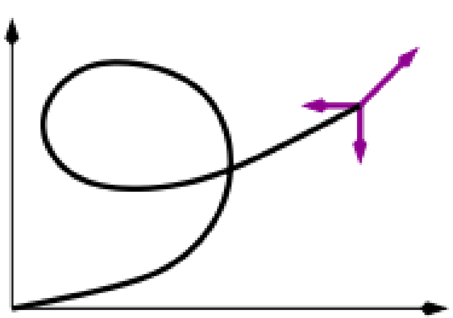
Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$

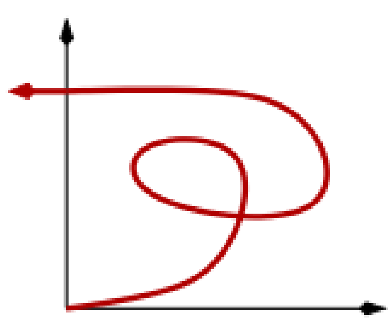
Write

$$Q(x,y) = \sum_{n\geq 0}\left(\sum_{i,j\geq 0} q_{i,j}(n)x^i y^j\right)t^n.$$

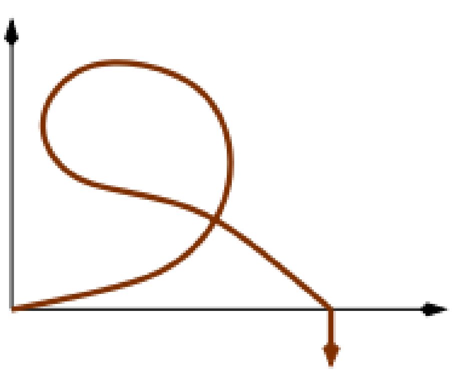
- Then $q_{i,j}(n)$ is the number of walks with n steps NE, W, S going from (0,0) to (i,j) in the first quadrant.



$\ominus$



$\ominus$



An equation for walks in a quadrant

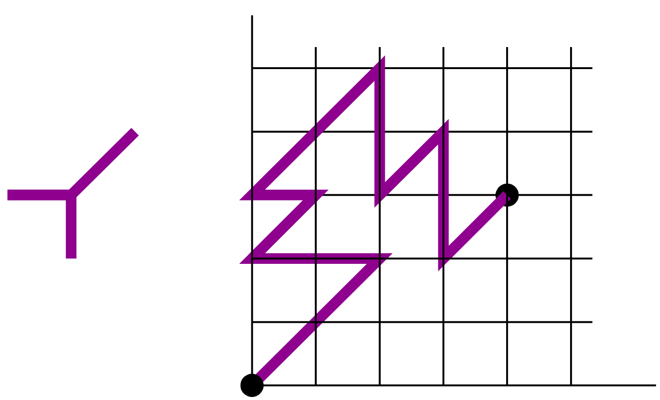
Series $Q(t;x,y) \equiv Q(x,y)$:

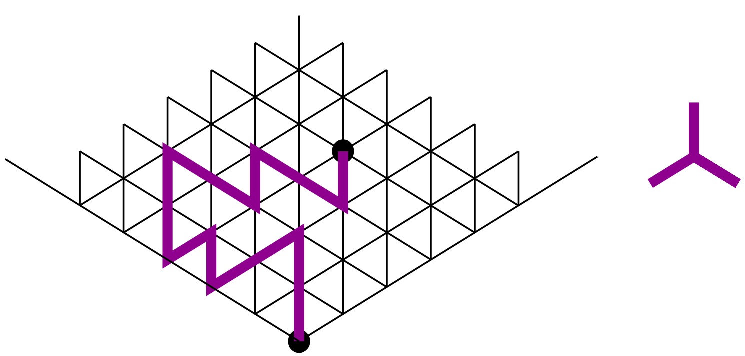
$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$

Write

$$Q(x,y) = \sum_{n\geq 0}\left(\sum_{i,j\geq 0} q_{i,j}(n)x^i y^j\right)t^n.$$

- Then $q_{i,j}(n)$ is the number of walks with n steps NE, W, S going from (0,0) to (i,j) in the first quadrant.





An equation for Kreweras' walks in a quadrant

Series $Q(t;x,y) \equiv Q(x,y)$:

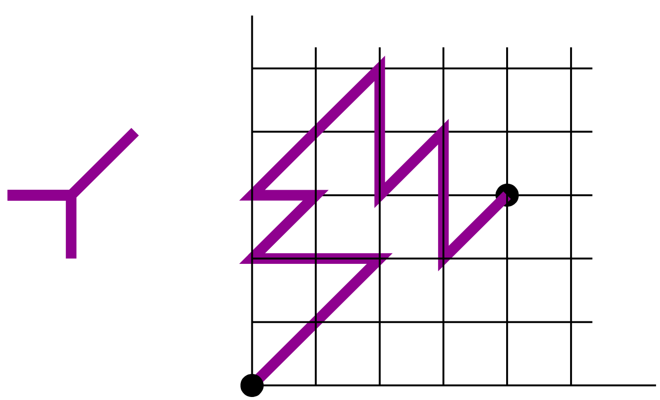
$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{\ }$$

Write

$$Q(x,y) = \sum_{n\geq 0} \left(\sum_{i,j\geq 0} q_{i,j}(n) x^i y^j \right) t^n.$$

- Then $q_{i,j}(n)$ is the number of walks with n steps NE, W, S, going from (0,0) to (i,j) in the first quadrant [Kreweras 65].

$$q_{i,0}(3n+2i) = \frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)} \binom{2i}{i} \binom{3n+2i}{n}$$



An equation for Kreweras' walks in a quadrant

Series $Q(t;x,y) \equiv Q(x,y)$:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{\;}$$

Write

$$Q(x,y) = \sum_{n\geq 0} \left(\sum_{i,j\geq 0} q_{i,j}(n) x^i y^j \right) t^n.$$

- Then $q_{i,j}(n)$ is the number of walks with n steps NE, W, S, going from (0,0) to (i,j) in the first quadrant [Kreweras 65].

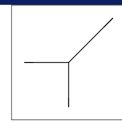
$$q_{i,0}(3n+2i) = \frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)} \binom{2i}{i} \binom{3n+2i}{n}$$

- The series $Q(t;x,y)$ is algebraic ! [Gessel 86]: there exists a non-zero polynomial $P(u,t,x,y)$ such that

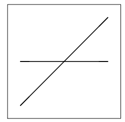
$$P(Q(t;x,y),t,x,y) = 0.$$

II. More equations in two catalytic variables

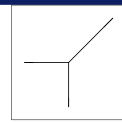
Quadrant walks with different steps

- Kreweras' walks 

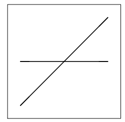
$$(1 - t\,(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- Gessel's walks 

Quadrant walks with different steps

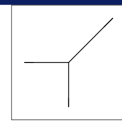
- Kreweras' walks 

$$(1 - t\,(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

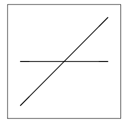
- Gessel's walks 

$$(1 - t(xy + \bar{x}\bar{y} + x + \bar{x}))xyQ(x,y) = xy - tQ(x,0) - t(1+y)Q(0,y) + tQ(0,0$$

Quadrant walks with different steps

- Kreweras' walks 

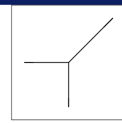
$$(1 - t\,(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- Gessel's walks 

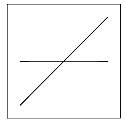
$$(1 - t(xy + \bar{x}\bar{y} + x + \bar{x}))xyQ(x,y) = xy - tQ(x,0) - t(1+y)Q(0,y) + tQ(0,0$$

- The kernel describes the new steps
- Coefficients on the r.h.s. have changed as well

Quadrant walks with different steps

- Kreweras' walks 

$$(1 - t(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- Gessel's walks 

$$(1 - t(xy + \bar{x}\bar{y} + x + \bar{x}))xyQ(x,y) = xy - tQ(x,0) - t(1+y)Q(0,y) + tQ(0,0$$

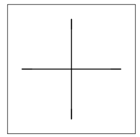
- The kernel describes the new steps
- Coefficients on the r.h.s. have changed as well

- This series is algebraic again ! [Bostan, Kauers 10]
- Some nice coefficients [Kauers, Koutschan, Zeilberger 09]

$$q_{0,0}(2n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Quadrant walks with different steps

- The simple walk 

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

- The kernel describes the new steps.
- Coefficients on the r.h.s. have changed as well

- The series Q(t;x,y) is not algebraic, but still D-finite.
- Nice coefficients:

$$q_{i,j}(n) = \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j}{2}+1}$$

A hierarchy of formal power series

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

A hierarchy of formal power series

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

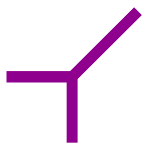
- D-algebraic

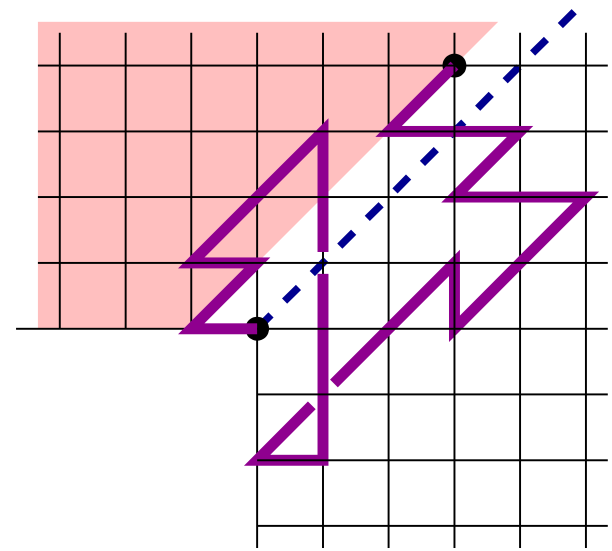
$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

Kreweras' walks in three quadrants

- Walks in a three-quadrant cone, ending above the diagonal:

$$2(1 - t(\bar{x}\bar{y} + x + y))xyU(x,y) = y - 2tU(x,0) + (ty + 2tx - 1)yD(y)$$



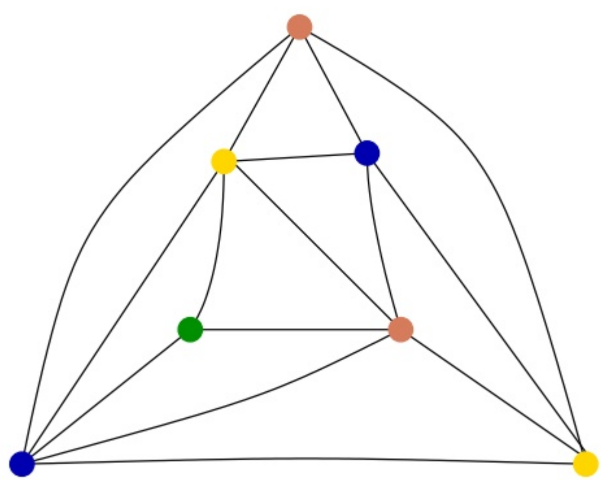


$$-2tU(0,y) = y - 2tU(0,0) + (ty - 1)yD(y)$$

Coloured triangulations: a historical example

Properly q-coloured triangulations : series T(t,q;x,y)≡T(x,y):

$$T(x,y) = x(q-1) + txyT(1,y)T(x,y) + tx\frac{T(x,y)-T(x,0)}{y} - tx^2y\frac{T(x,y)-T(1,y)}{x-1}$$



- Known : T(1,0) is D-algebraic, and algebraic for some q (q=2, q=3...)

[Tutte, 1973-1984]

The birth of invariants

Three-stack sortable permutations

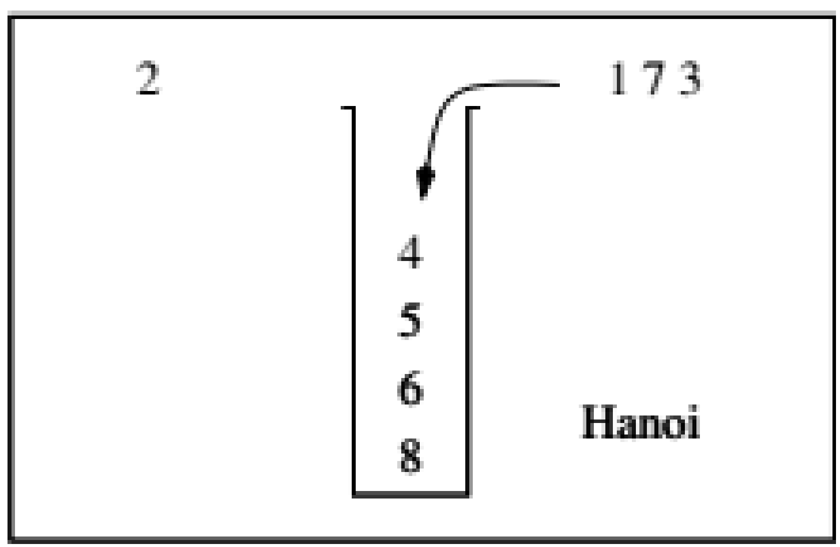
- A non-linear example [Defant, Elvey Price, Guttmann 21]

$$P(x,y) = t(x+1)^2(y+1)^2 + ty(1+x)P(x,y) + t(1+x)\frac{P(x,y) - P(x,0)}{y}\left((1+y)^2 + y\frac{P(x,y) - P(0,y)}{x}\right)$$

Stack-sorting [Knuth 68]

Not D-finite?

$S(\sigma)$



σ

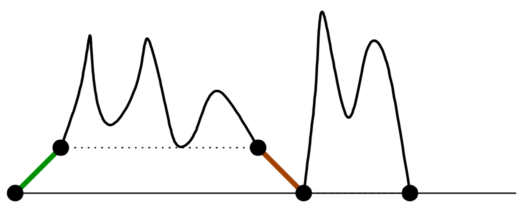
$S^3(\sigma) = 12\cdots n$

III. Equations with catalytic variables: 0<1<2<...

No catalytic variable: polynomial equations

- Example 1: Dyck paths, counted by steps

$$D = 1 + t^2 D^2$$



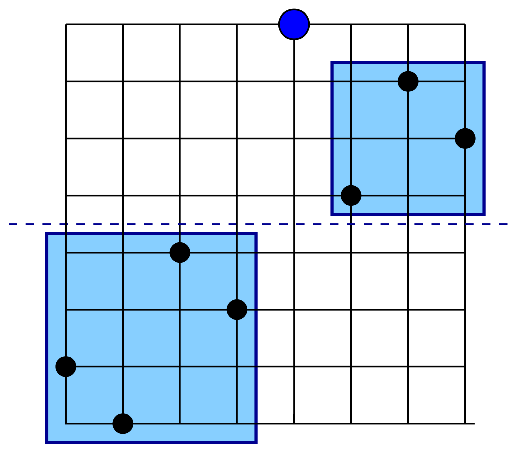
- Example 2: plane trees with degrees 5 and 18

$$T = 1 + t\, T^5 + t\, T^{18}$$

- Many branching structures

- One-stack sortable permutations

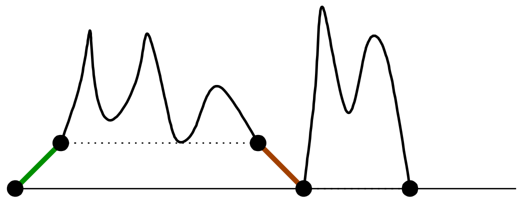
$$P = 1 + t\, P^2$$



No catalytic variable: polynomial equations

- Example 1: Dyck paths, counted by steps

$$D = 1 + t^2 D^2$$



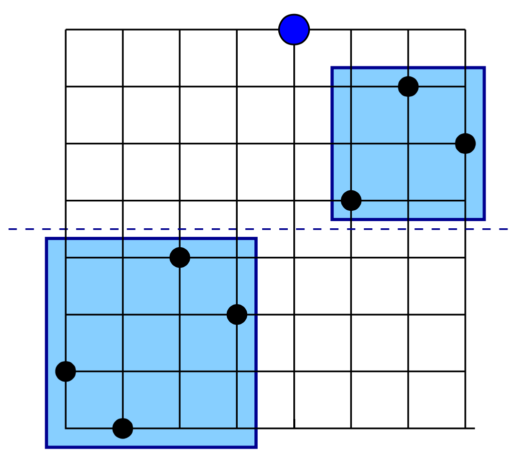
- Example 2: plane trees with degrees 5 and 18

$$T = 1 + t\,T^5 + t\,T^{18}$$

- Many branching structures

- One-stack sortable permutations

$$P = 1 + t\,P^2$$



Algebraic series

One catalytic variable x

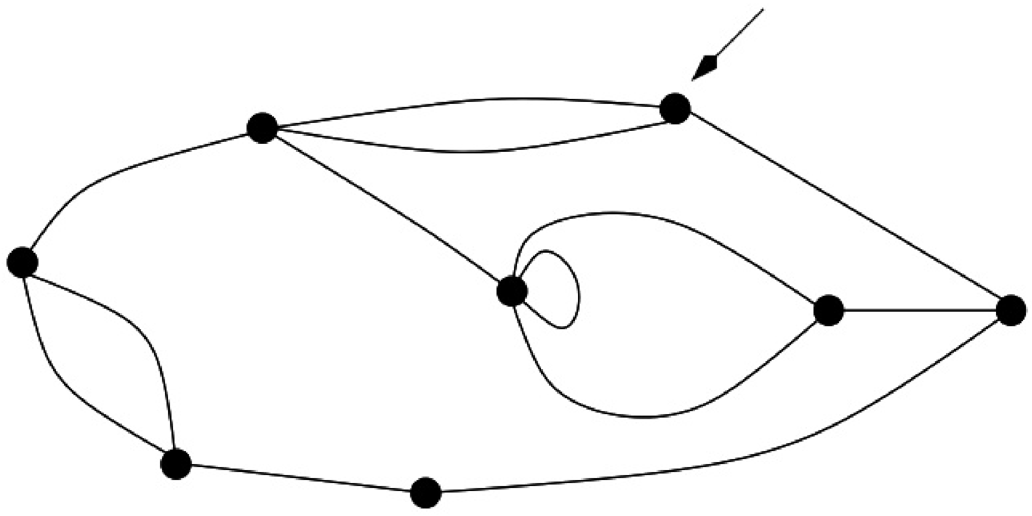
- Planar maps [Tutte 68]

$$M(x) = 1 + tx^2M(x)^2 + tx\frac{xM(x) - M(1)}{x-1}$$

or, with A(x)=M(x+1):

$$A(x) = 1 + t(x+1)^2A(x)^2 + t(x+1)\frac{(x+1)A(x) - A(0)}{x}$$

- Many families of (uncoloured) maps



One catalytic variable x

- Two-stack sortable permutations [Zeilberger 92]

$$A(x) = \frac{1}{1-tx} + xt\,\frac{xA(x) - A(1)}{x-1} \cdot \frac{A(x) - A(1)}{x-1}$$

One catalytic variable x

- Two-stack sortable permutations [Zeilberger 92]

$$A(x) = \frac{1}{1-tx} + xt\,\frac{xA(x)-A(1)}{x-1}\cdot\frac{A(x)-A(1)}{x-1}$$

Theorem [MBM-Jehanne 06]
Let $P(A(x), A_1, A_2, \ldots, A_k, t, x)$ be a polynomial equation in one catalytic variable x. Under natural assumptions, the series A(x) and the A_i's are algebraic.

Algebraic series

One catalytic variable x

- Two-stack sortable permutations [Zeilberger 92]

$$A(x) = \frac{1}{1-tx} + xt\,\frac{xA(x) - A(1)}{x-1} \cdot \frac{A(x) - A(1)}{x-1}$$

Theorem [MBM-Jehanne 06]
Let $P(A(x), A_1, A_2, \ldots, A_k, t, x)$ be a polynomial equation in one catalytic variable x. Under natural assumptions, the series A(x) and the A_i's are algebraic.

[Popescu 85, Swan 98]

Algebraic series

One catalytic variable x

- Two-stack sortable permutations [Zeilberger 92]

$$A(x) = \frac{1}{1-tx} + xt\,\frac{xA(x) - A(1)}{x-1} \cdot \frac{A(x) - A(1)}{x-1}$$

Theorem [MBM-Jehanne 06]
Let $P(A(x), A_1, A_2, \ldots, A_k, t, x)$ be a polynomial equation in one catalytic variable x. Under natural assumptions, the series A(x) and the A_i's are algebraic.

[Popescu 85, Swan 98]

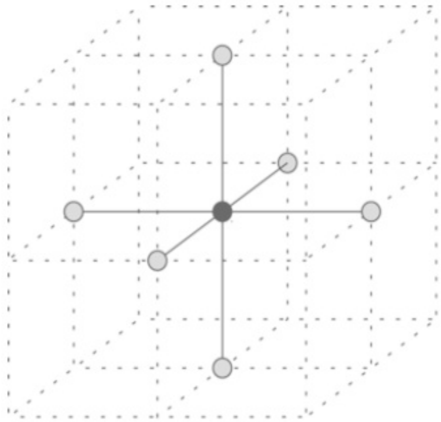
effective

Algebraic series

Arbitrarily many catalytic variables

- Walks in $\mathbb{N}^d$ with unit steps $(0,\ldots, 0, \pm 1, 0, \ldots, 0)$:

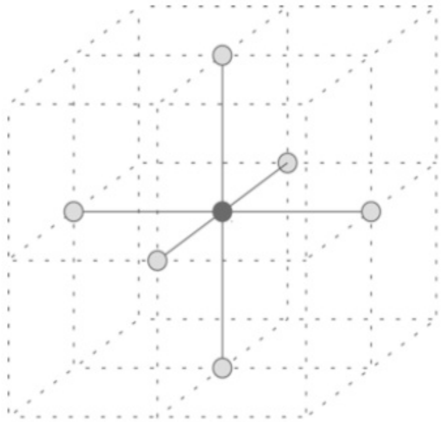
  D-finite



Arbitrarily many catalytic variables

- Walks in $\mathbb{N}^d$ with unit steps (0,..., 0, ± 1, 0, ..., 0) :

  D-finite



- Permutations with no ascending sub-sequence of length (d+2)

  [mbm 11] D-finite [Gessel 90]

- Permutations sortable by (d+1) stacks ?

IV. Back to two catalytic variables: Tutte’s invariants

A tool for proving (D)-algebraicity

Framework

An equation in two catalytic variables, defining a series A(t;x,y):

- linear in A(t;x,y)≡A(x,y),
- with two divided differences at x=0 and y=0, of first order.

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the kernel K(x,y) satisfies:

$$K(x,y) = 1 - \bar{x}\bar{y}tS(t,x,y,A(x,0),A(0,y))$$

for polynomials R and S.

Framework

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the kernel K(x,y) satisfies:

$$K(x,y) = 1 - t\bar{x}\bar{y}S(t,x,y,A(x,0),A(0,y))$$

for polynomials R and S.

Framework

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the kernel K(x,y) satisfies:

$$K(x,y) = 1 - t\bar{x}\bar{y}S(t,x,y,A(x,0),A(0,y))$$

for polynomials R and S.

Example 1: quadrant walks with Kreweras' steps

$$(1 - t\,(xy + \bar{x} + \bar{y}))xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

Here,

$$K(x,y) = 1 - t\bar{x}\bar{y}(x^2y^2 + y + x).$$

Framework

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the kernel K(x,y) satisfies:

$$K(x,y) = 1 - t\bar{x}\bar{y}S(t,x,y,A(x,0),A(0,y))$$

for polynomials R and S.

Example 2: Tutte's coloured triangulations

$$T(u,y) = ux(q-1) + tuyT(1,y)T(u,y) + tu\frac{T(u,y)-T(u,0)}{y} - tu^2y\frac{T(u,y)-T(1,y)}{u-1}$$

Framework

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the kernel K(x,y) satisfies:

$$K(x,y) = 1 - t\bar{x}\bar{y}S(t,x,y,A(x,0),A(0,y))$$

for polynomials R and S.

Example 2: Tutte's coloured triangulations

$$T(u,y) = ux(q-1) + tuyT(1,y)T(u,y) + tu\frac{T(u,y)-T(u,0)}{y} - tu^2y\frac{T(u,y)-T(1,y)}{u-1}$$

Or, with A(x,y)=T(x+1,y) and u=x+1,

$$\left(1 - t\bar{x}\bar{y}\left(ux - u^2y^2 + uxy^2A(0,y)\right)\right)xyA(x,y) = xuy(q-1) - txuyA(x,0) + tu^2y^2A(0,y)$$

Framework

- In our equations,

$$\frac{1}{K(x,y)} = \frac{1}{1 - t\bar{x}\bar{y}S(t,x,y,A(x,0),A(0,y))} = \sum_{k\geq 0} t^k \bar{x}^k \bar{y}^k S(t,x,y,A(x,0),A(0,y))^k$$

has poles of unbounded order at x=0 and y=0.

Example: for simple walks in the quadrant,

$$\frac{1}{K(x,y)} = \frac{1}{1 - t(x+\bar{x}+y+\bar{y})} = \sum_{n\geq 0} (x+\bar{x}+y+\bar{y})^n t^n$$

and the n-th coefficient has a pole of order n at x=0 (and at y=0).

Invariants for a kernel K(x,y)

Def. A pair of series (I(x),J(y)), with rational coefficients in x (resp. y) is a pair of invariants if

$$(I(x)-J(y))/K(x,y)$$

has poles of bounded order (p.b.o.) at x=0 and y=0.

Invariants for a kernel K(x,y)

Def. A pair of series (I(x),J(y)), with rational coefficients in x (resp. y) is a pair of invariants if

(I(x)-J(y))/K(x,y)

has poles of bounded order (p.b.o.) at x=0 and y=0.

Poles of bounded order ?

$$\frac{1}{K(x,y)} = \frac{1}{1-t(x+\bar{x}+y+\bar{y})} = \sum_{n\geq 0}(x+\bar{x}+y+\bar{y})^n t^n$$ no

$$\sum_{n\geq 0}\frac{x^n}{(1-x^n)y^n}t^n$$ no

$$\sum_{n\geq 0}\frac{1+xy}{x^2y^5(1+x)^{2n}(1-5y)^n}t^n$$ yes

Explicit kernels, explicit invariants

Def. A pair of series $(I(x),J(y))$ is a pair of invariants if

$$(I(x)-J(y))/K(x,y)$$

has poles of bounded order at $x=0$ and $y=0$.

Explicit kernels, explicit invariants

Def. A pair of series (I(x),J(y)) is a pair of invariants if

(I(x)-J(y))/K(x,y)

has poles of bounded order at x=0 and y=0.

Example 1. Simple walks in the quadrant :

$$K(x,y) = 1 - t(x+\bar{x}+y+\bar{y}) = (1-t(x+\bar{x})) - t(y+\bar{y})$$

Then

$$I_0(x) := 1 - t(x+\bar{x}) \quad \text{and} \quad J_0(y) := t(y+\bar{y})$$

form a pair of invariants since

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = 1.$$

Explicit kernels, explicit invariants

Def. A pair of series (I(x),J(y)) is a pair of invariants if

$$(I(x)-J(y))/K(x,y)$$

has poles of bounded order at x=0 and y=0.

Example 2. Kreweras' walks in the quadrant :

$$K(x,y) = 1 - t(xy + \bar{x} + \bar{y})$$

Then

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x, \qquad J_0(y) = I_0(y)$$

form a pair of invariants since

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t}$$

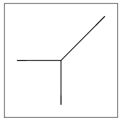
From a functional equation to invariants

Def. A pair of series (I(x),J(y)) is a pair of invariants if

$$(I(x)-J(y))/K(x,y)$$

has poles of bounded order at x=0 and y=0.

Example 2. Kreweras' walks in the quadrant:



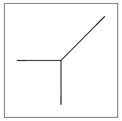
$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

From a functional equation to invariants

Def. A pair of series (I(x),J(y)) is a pair of invariants if

$$(I(x)-J(y))/K(x,y)$$

has poles of bounded order at x=0 and y=0.



Example 2. Kreweras' walks in the quadrant:

$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

Since

$$xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{K(x,y)}{t},$$

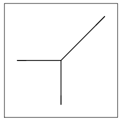
From a functional equation to invariants

Def. A pair of series (I(x),J(y)) is a pair of invariants if

(I(x)-J(y))/K(x,y)

has poles of bounded order at x=0 and y=0.

Example 2. Kreweras' walks in the quadrant:



$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

Since

$$xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{K(x,y)}{t},$$

we have

$$K(x,y)\left(xyQ(x,y) + \frac{1}{t}\right) = I_1(x) - J_1(y)$$

and a second pair of invariants:

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0), \qquad J_1(y) = -I_1(y).$$

The invariant lemma

Lemma. Let $(I(x),J(y))$ be a pair of invariants such that the series

$$(I(x)-J(y))/K(x,y)$$

not only has p.b.o. at $x=0$ and $y=0$, but in fact vanishes at $x=0$ and $y=0$ ("strict" invariants). Then $I(x)$ and $J(y)$ are trivial:

$$I(x)=J(y) \text{ is independent of } x \text{ (and } y).$$

Lemma. The componentwise sum and product of two pairs of invariants $(I_0(x), J_0(y))$, $(I_1(x), J_1(y))$ is another pair of invariants.

The invariant lemma

Lemma. Let (I(x),J(y)) be a pair of invariants such that the series

$$(I(x)-J(y))/K(x,y)$$

not only has p.b.o. at x=0 and y=0, but in fact vanishes at x=0 and y=0 ("strict" invariants). Then I(x) and J(y) are trivial:

I(x)=J(y) is independent of x (and y).

$$\frac{I(x)-J(y)}{K(x,y)} = \sum_n \frac{xy\, p_n(x,y)}{d_n(x)d'_n(y)} t^n$$

Lemma. The componentwise sum and product of two pairs of invariants $(I_0(x), J_0(y))$, $(I_1(x), J_1(y))$ is another pair of invariants.

Application: Kreweras' walks in the quadrant

- Two pairs of invariants:

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x)$$

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t}$$

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0) = -J_1(x)$$

$$\frac{I_1(x) - J_1(y)}{K(x,y)} = xyQ(x,y) + \frac{1}{t}$$

Application: Kreweras' walks in the quadrant

- Two pairs of invariants:

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x)$$

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t}$$

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0) = -J_1(x)$$

$$\frac{I_1(x) - J_1(y)}{K(x,y)} = xyQ(x,y) + \frac{1}{t}$$

- Observation: the following pair of invariants has no pole at x=0 or y=0:

$$I(x) := I_1(x)^2 - I_0(x), \qquad J(y) := J_1(y)^2 - J_0(y)$$

Application: Kreweras' walks in the quadrant

- **Two pairs of invariants:**

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x) \qquad I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0) = -J_1(x)$$

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t} \qquad \frac{I_1(x) - J_1(y)}{K(x,y)} = xyQ(x,y) + \frac{1}{t}$$

- **Observation:** the following pair of invariants has no pole at x=0 or y=0:

$$I(x) := I_1(x)^2 - I_0(x), \qquad J(y) := J_1(y)^2 - J_0(y)$$

Moreover, (I(x)-J(y))/K(x,y) vanishes at x=0 and y=0!

Application: Kreweras' walks in the quadrant

- **Two pairs of invariants:**

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x) \qquad I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x,0) = -J_1(x)$$

$$\frac{I_0(x) - J_0(y)}{K(x,y)} = \frac{x-y}{x^2y^2} \cdot \frac{1}{t} \qquad \frac{I_1(x) - J_1(y)}{K(x,y)} = xyQ(x,y) + \frac{1}{t}$$

- **Observation:** the following pair of invariants has no pole at x=0 or y=0:

$$I(x) := I_1(x)^2 - I_0(x), \qquad J(y) := J_1(y)^2 - J_0(y)$$

Moreover, (I(x)-J(y))/K(x,y) vanishes at x=0 and y=0!
By the invariant lemma, I(x)=I(0), that is,

$$(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = -1/(2t)^2 + 2tQ(0,0)$$

Application: Kreweras' walks in the quadrant

Summary: starting from an equation in two catalytic variables, involving $Q(x,y)$, $Q(x,0)$ and $Q(0,y)$, we have derived an equation in only one catalytic variable, involving $Q(x,0)$ and $Q(0,0)$ only

⇒ algebraicity

$$(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = -1/(2t)^2 + 2tQ(0,0)$$

Application: Kreweras' walks in the quadrant

Summary: starting from an equation in two catalytic variables, involving Q(x,y), Q(x,0) and Q(0,y), we have derived an equation in only one catalytic variable, involving Q(x,0) and Q(0,0) only

⇒ algebraicity

$$(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = -1/(2t)^2 + 2tQ(0,0)$$

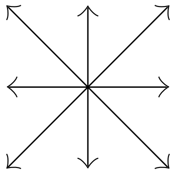
Theorem [Kreweras 65, Gessel 86, mbm 05...]: let Z(t) ≡ Z be the only series in t such that $Z=t(2+Z^3)$. Then

$$Q(x,0) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{Z} - \frac{1}{x}\right)\sqrt{1 - xZ^2}\right).$$

V. Walks in a quadrant: the whole picture

About twenty years ago...

- Systematic study of quadrant walks

Set of steps ("model") in 

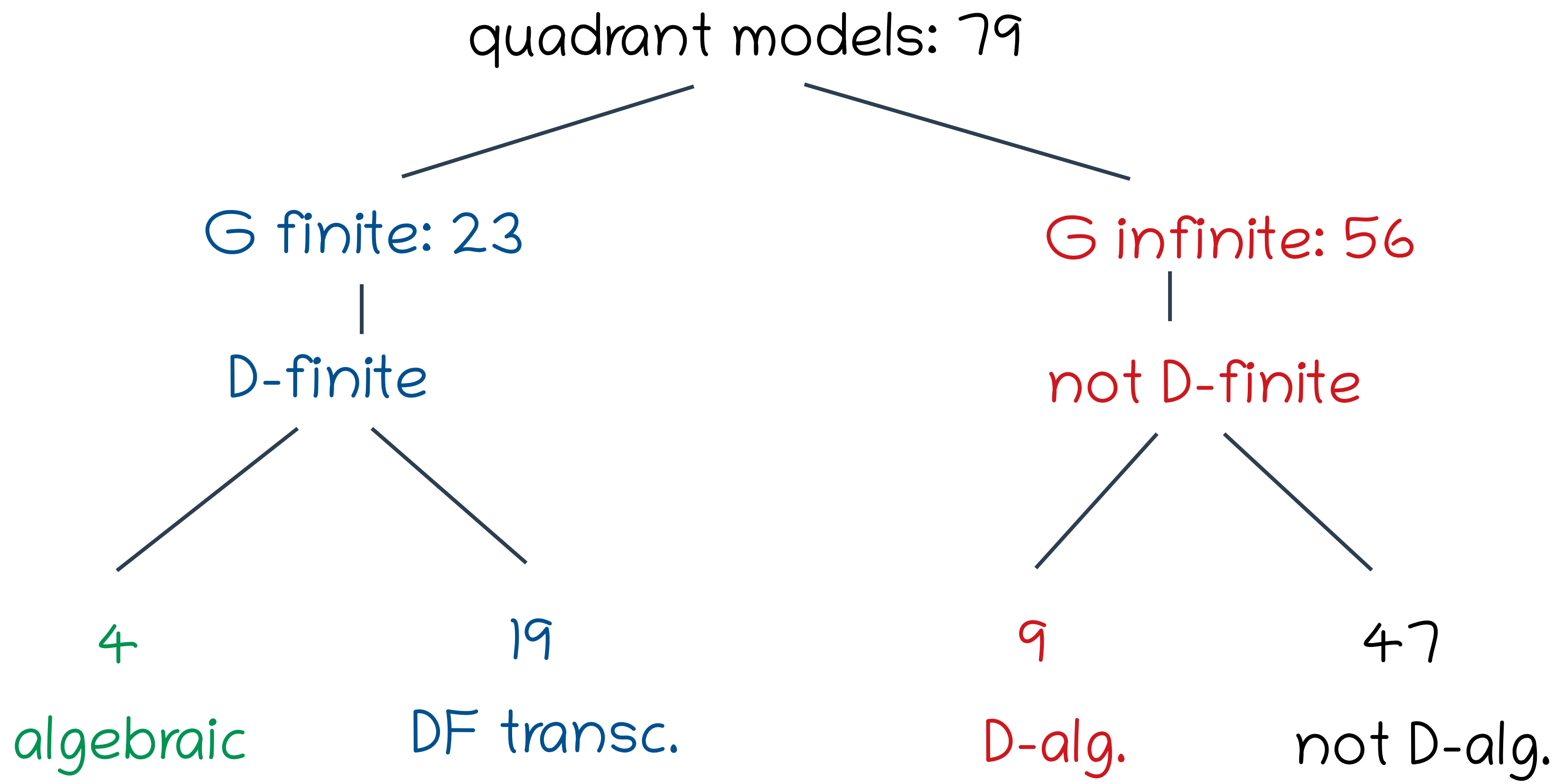
- Some models are trivial, or equivalent to a half plane problem

$\Rightarrow$ 79 really interesting and distinct small step models [mbm-Mishna 10]

- Systematic approach via a functional equation

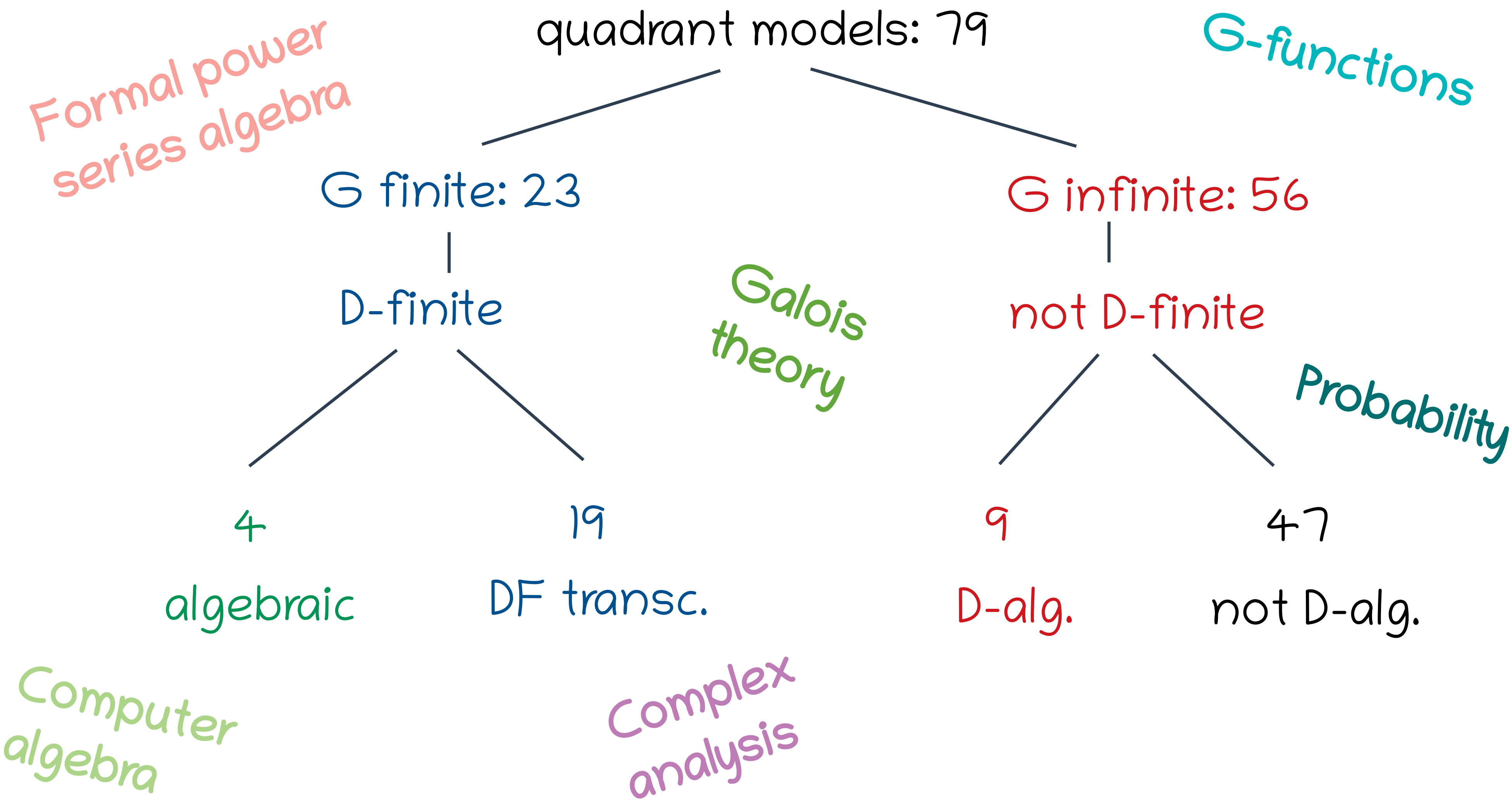
$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

Twenty years later: classification of quadrant walks



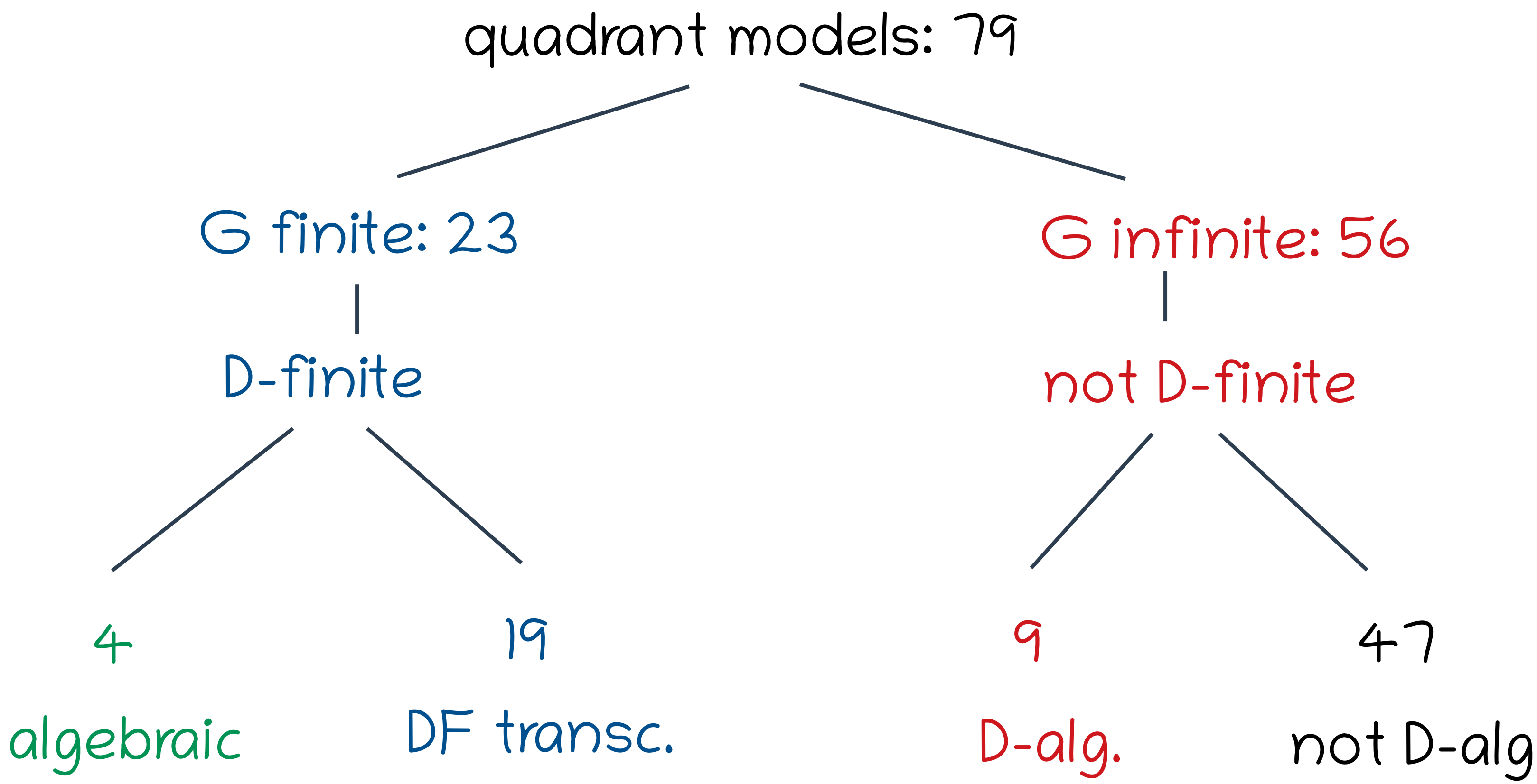
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

Twenty years later: classification of quadrant walks



Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

Twenty years later: classification of quadrant walks



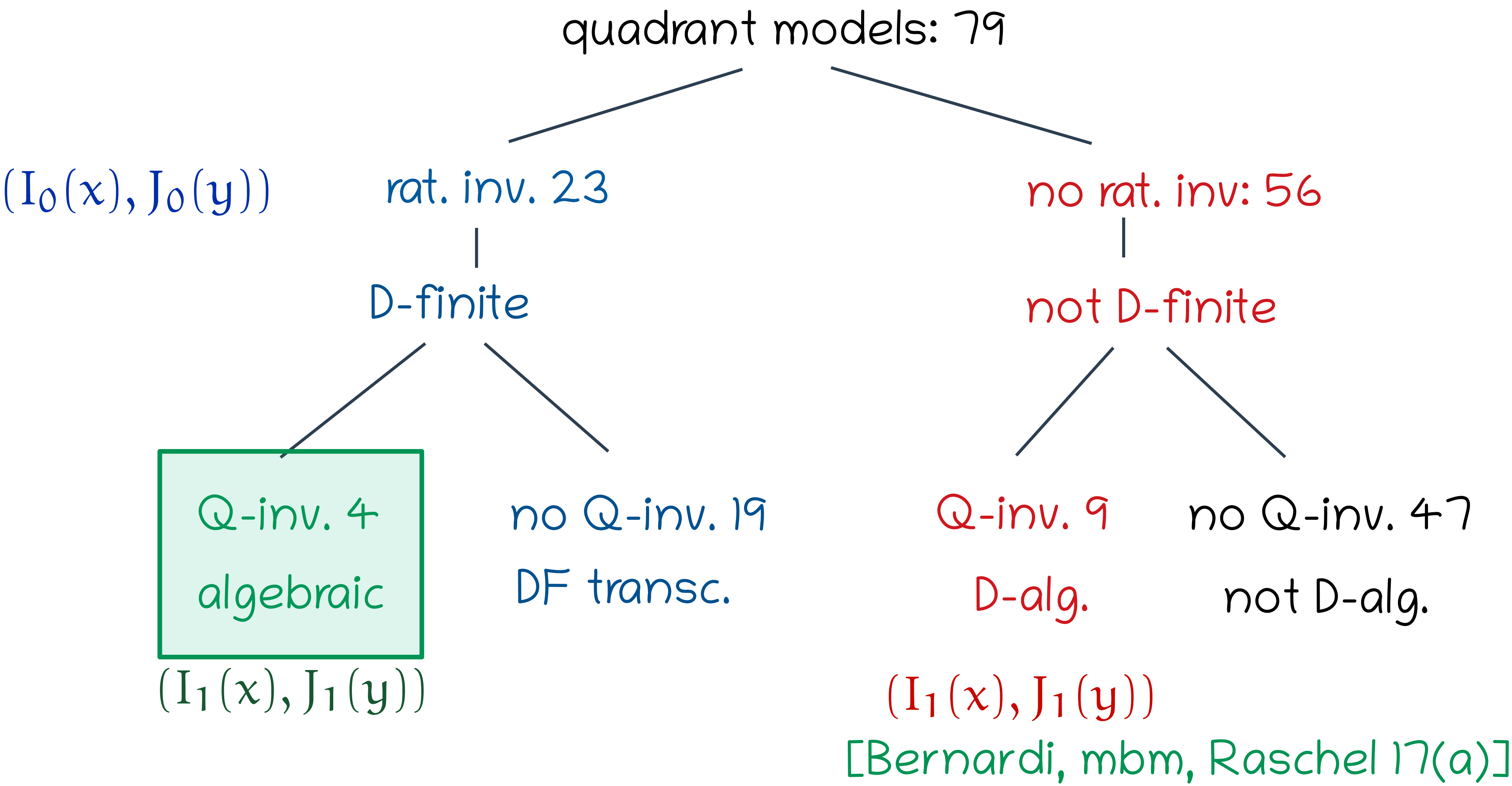
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

Twenty years later: classification of quadrant walks

- quadrant models: 79
  - rat. inv. 23 — $(I_0(x), J_0(y))$
    - D-finite
      - Q-inv. 4 — algebraic — $(I_1(x), J_1(y))$
      - no Q-inv. 19 — DF transc.
  - no rat. inv: 56
    - not D-finite
      - Q-inv. 9 — D-alg. — $(I_1(x), J_1(y))$
      - no Q-inv. 47 — not D-alg.

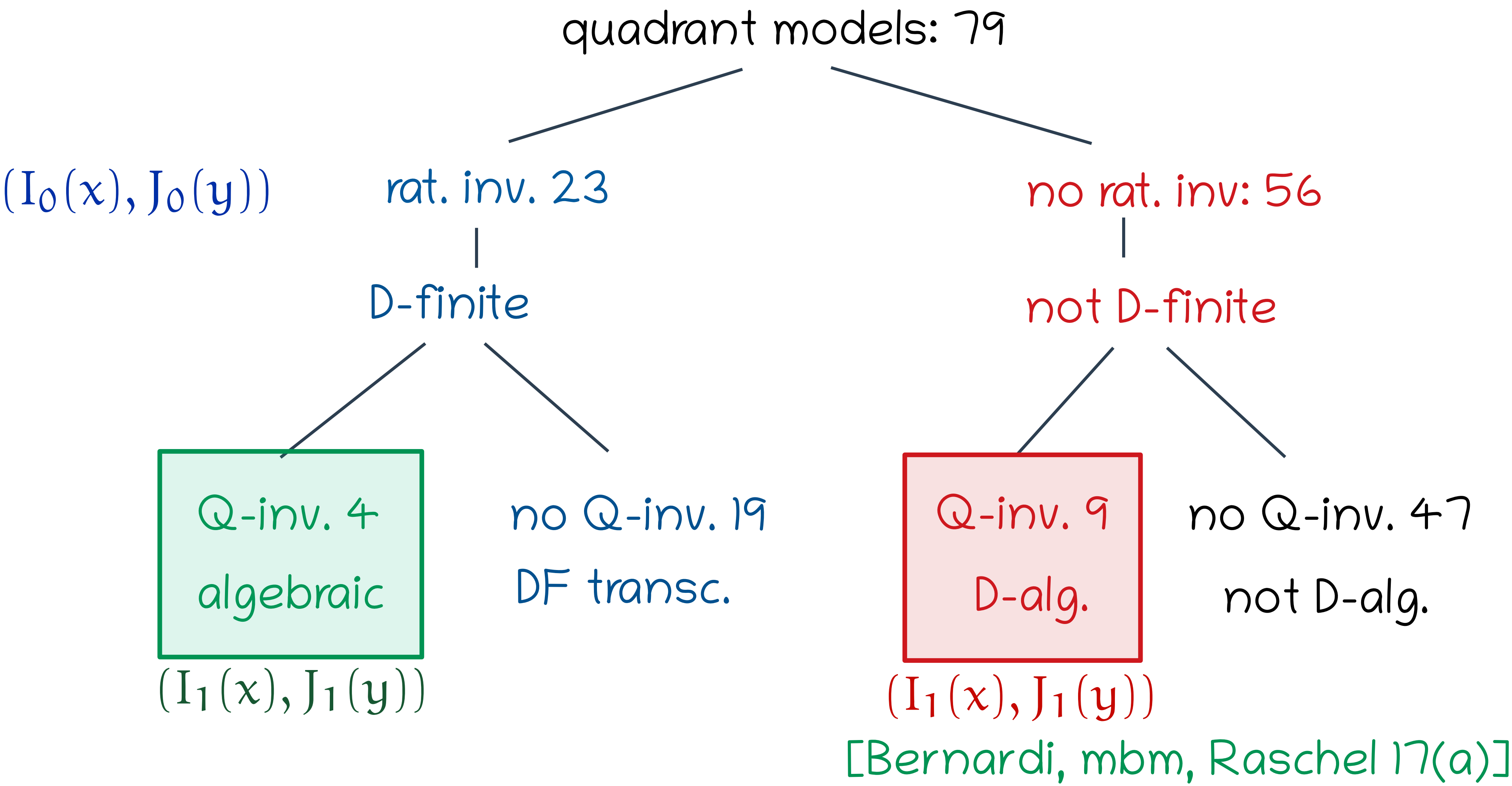
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

Twenty years later: classification of quadrant walks



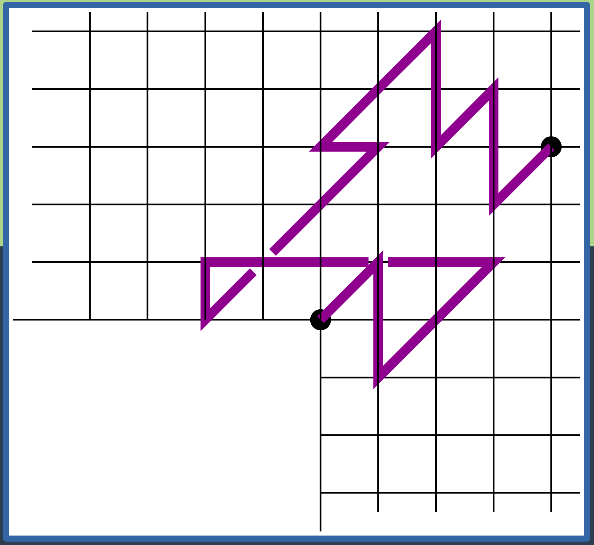
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

Twenty years later: classification of quadrant walks



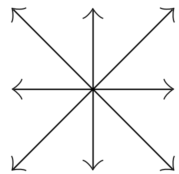
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

VI. Walks in three quadrants: a partial picture



Since 2015...

- Systematic study of three-quadrant walks with small steps

Set of steps in 

- Some models are trivial, or equivalent to a half plane problem

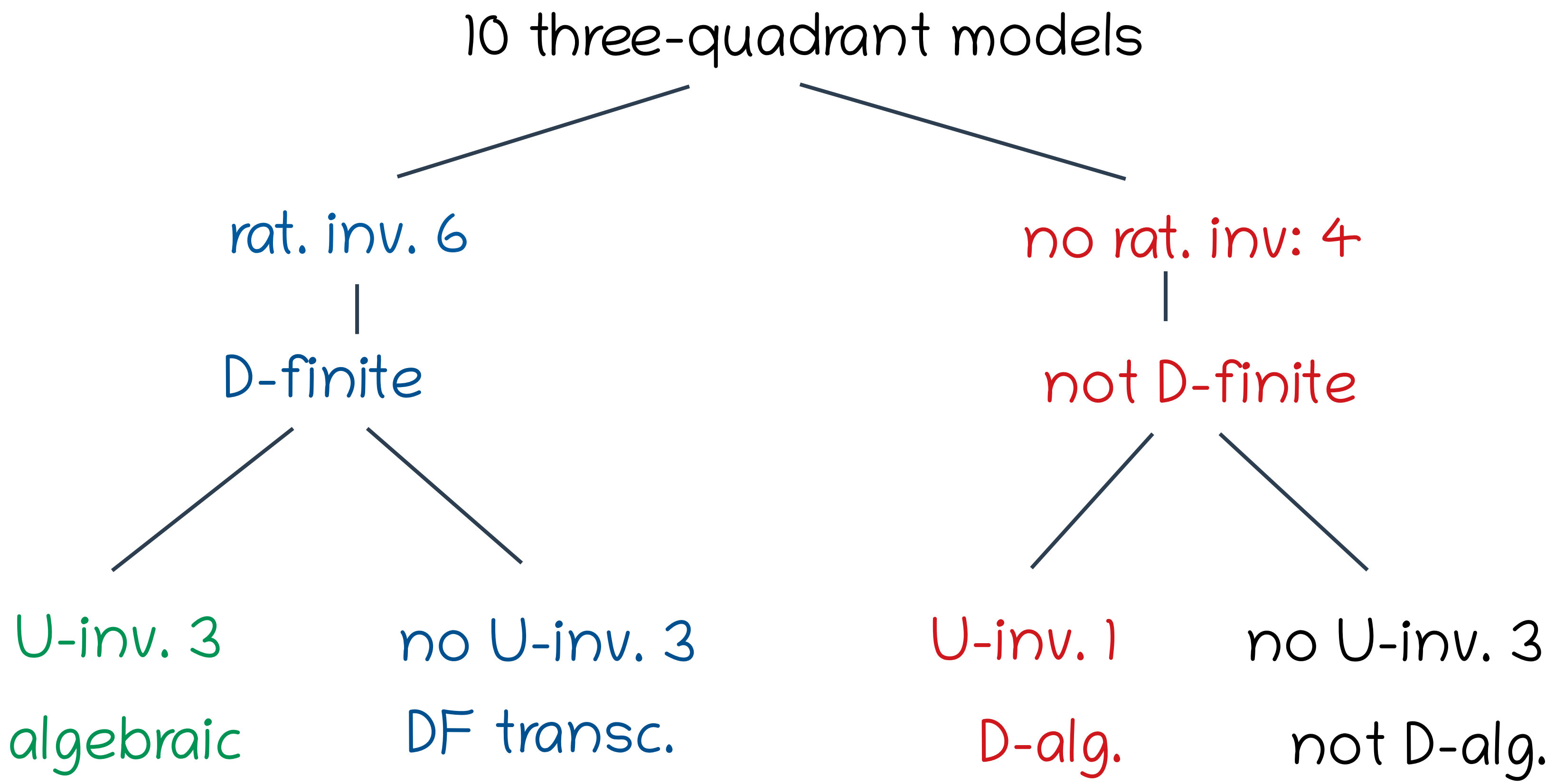
$\Rightarrow$ 74 really interesting and distinct models

- For ten x/y-symmetric step sets*, an equation reminiscent of quadrant equations:

$$2(1 - t(\bar{x}\bar{y} + x + y))xyU(x, y) = y - 2tU(x, 0) + (ty + 2tx - 1)yD(y)$$

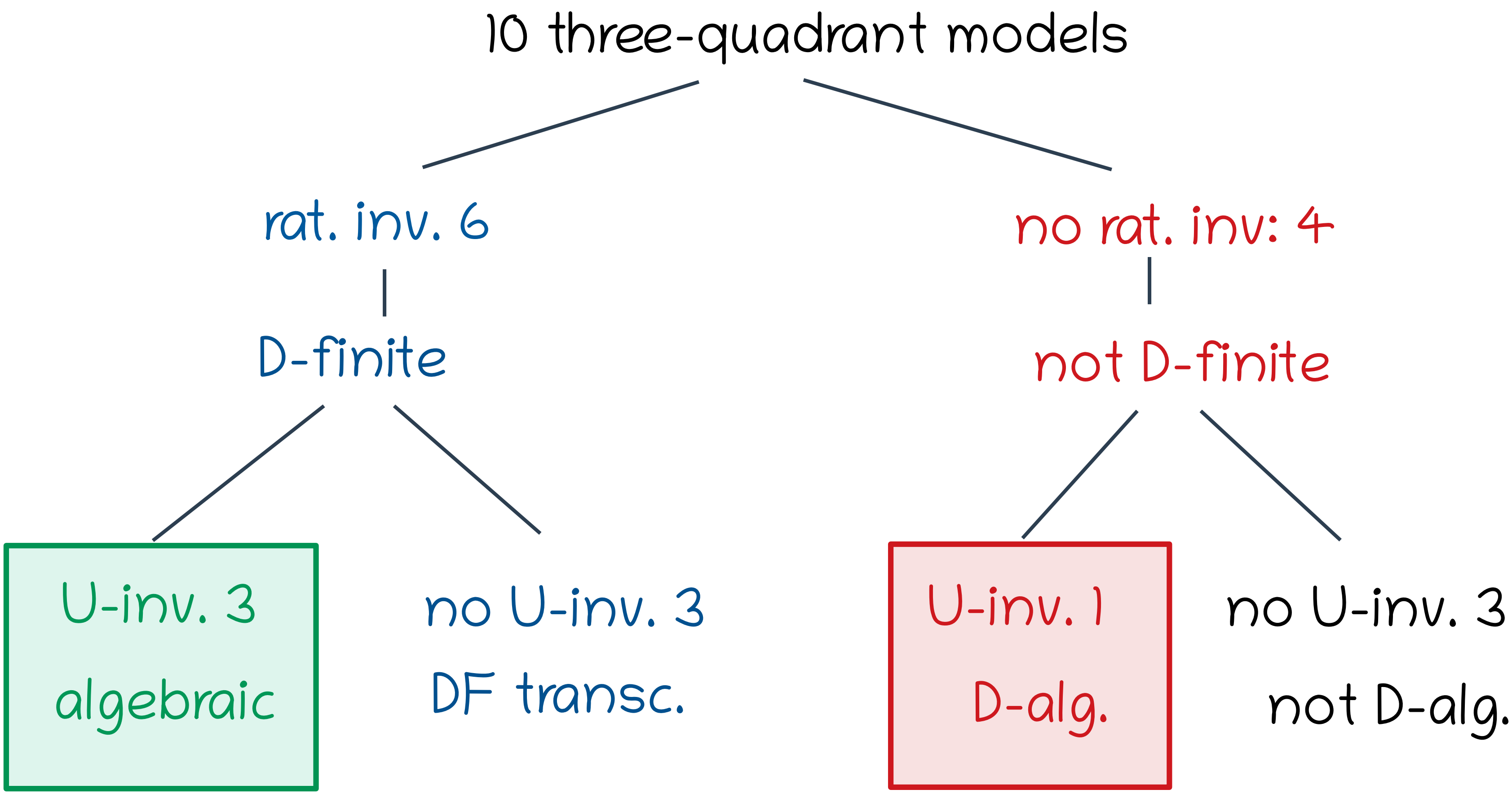
* those with no NW nor SE step

A partial classification of three-quadrant walks



mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

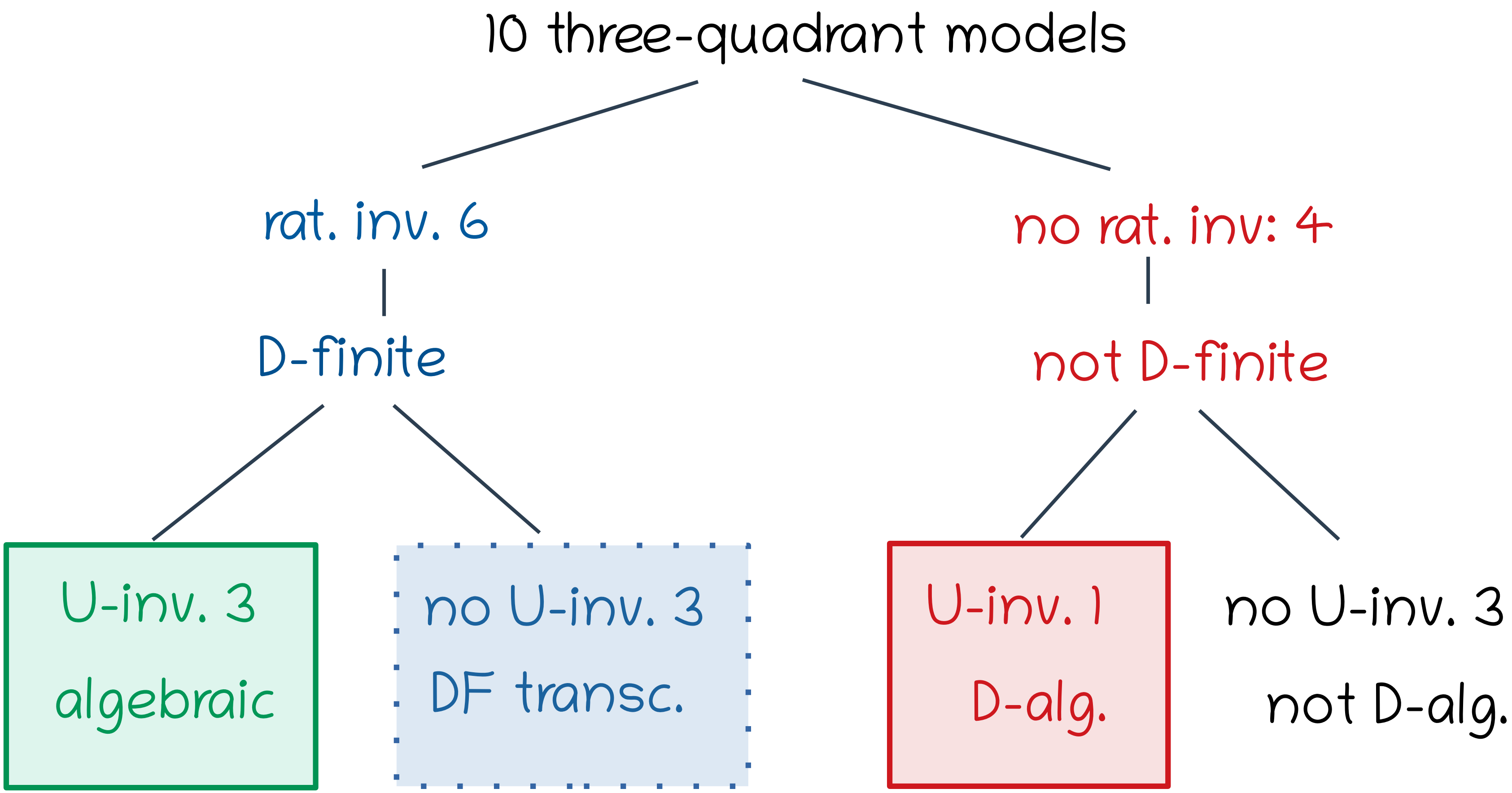
A partial classification of three-quadrant walks



[mbm 21(a)]

mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

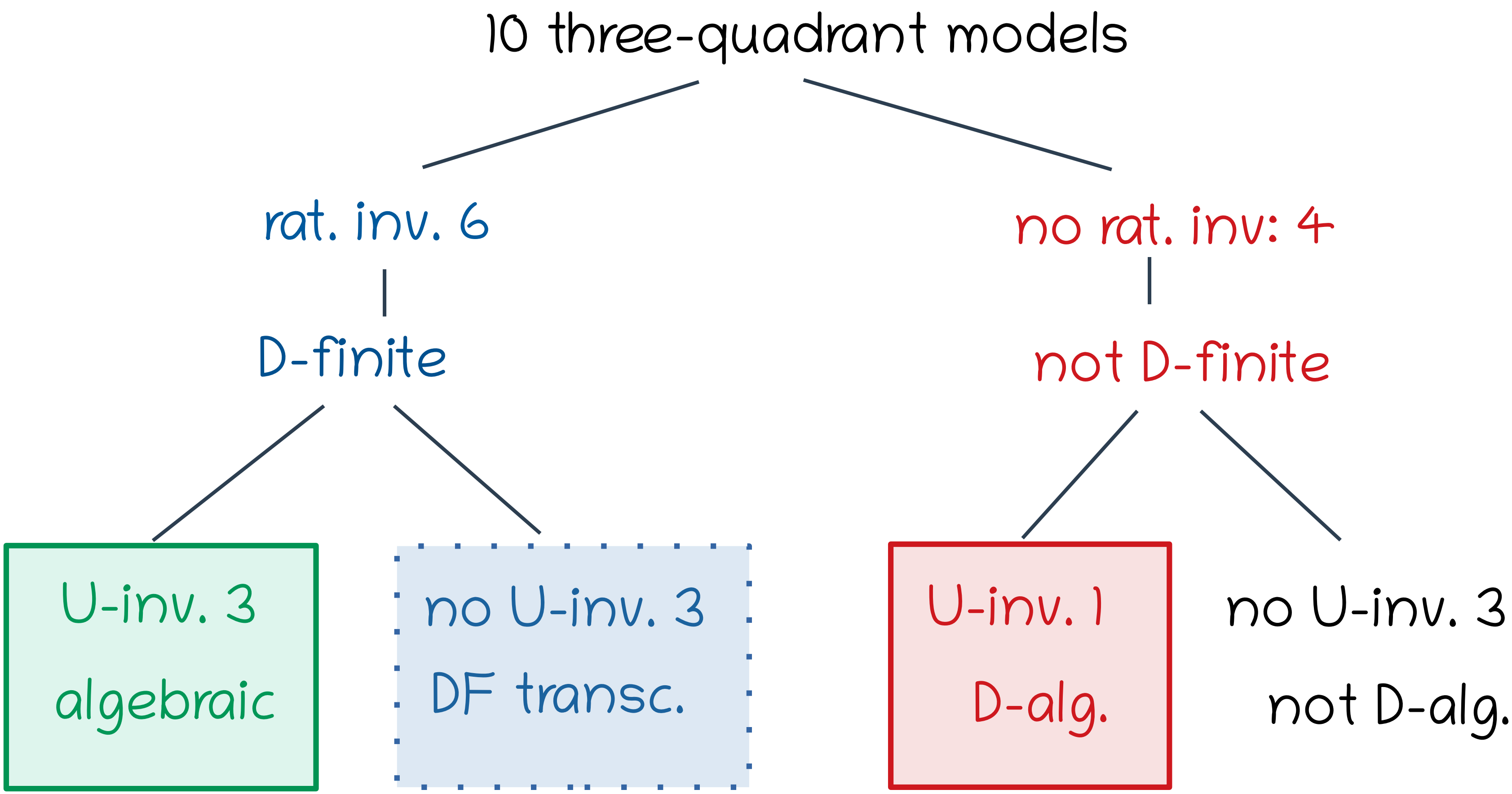
A partial classification of three-quadrant walks



[mbm 21(a)]

mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

A partial classification of three-quadrant walks

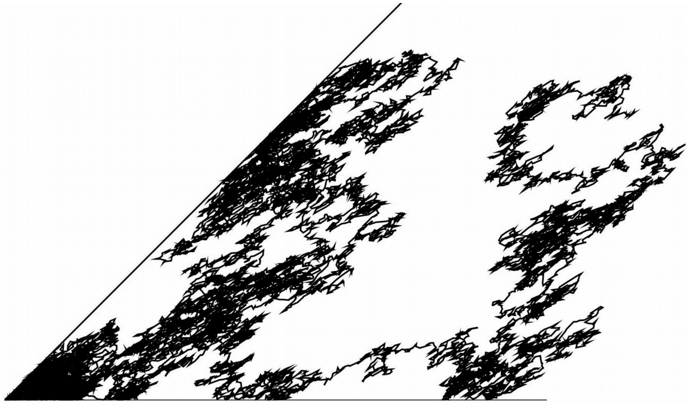


[mbm 21(a)]

Theorem [Elvey Price 22(a)] The Gfs of quadrant walks and three quadrant walks with the same (small) steps are of the same nature, at least w.r.t. x and y.

Applications of Tutte's invariants

- Properly coloured triangulations [Tutte 73-84]
- General colourings of maps (= Potts model) [Bernardi-mbm 11-17]
- Quadrant walks [Bernardi, mbm, Raschel 17(a)]
- Three-quadrant walks [mbm 21(a)]
- Continuous walks in a cone [mbm, Elvey Price, Franceschi, Hardouin, Raschel...]



Perspectives/work in progress

- (D)-algebraicity for more three-quadrant walks, e.g. Gessel's walks
- Quadrant walks with larger steps (P. Bonnet)
- 3-dimensional walks: from 3 to 2 catalytic variables?

Perspectives/work in progress

- (D)-algebraicity for more three-quadrant walks, e.g. Gessel's walks
- Quadrant walks with larger steps (P. Bonnet)
- 3-dimensional walks: from 3 to 2 catalytic variables?

Constructing invariants,

- from an explicit rational kernel
- or from a functional equation

should be automatized (if possible...).

[Buchacher, Kauers, Pogudin 20(a)]