The calculus of series-divisors

Hartosh Singh Bal and Gaurav Bhatnagar*

“The partition-frequency enumeration matrix” (Ramanujan J. (to appear))

“Glaisher’s divisors and infinite products” (preprint)

Ramanujan's recurrence relation

$$\sum_{d=1}^{n} \sigma(d)p(n-d) = np(n)$$

Berndt: Part IV

$$\sigma(n) = \sum_{d|n} d$$

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \sum_{n=0}^{\infty} p(n)q^n$$

J W Glaisher (1885)

$$d_1(n) = \sum_{d|n} d = \sigma(n)$$

$$d_2(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d = \sigma(n) - 2\sigma(n/2)$$

$$d_3(n) = \sum_{\substack{d|n \\ d \text{ even}}} d = 2\sigma(n/2)$$

$$d_4(n) = \sum_{\substack{d|n \\ n/d \text{ odd}}} d = \sigma(n) - \sigma(n/2)$$

$$d_5(n) = \sum_{\substack{d|n \\ n/d \text{ even}}} d = \sigma(n/2)$$

$$d_6(n) = \sum_{d|n} (-1)^{d-1} d = \sigma(n) - 4\sigma(n/2)$$

$$d_7(n) = \sum_{d|n} (-1)^{n/d-1} d = \sigma(n) - 2\sigma(n/2)$$

Glaisher's Divisors

Questions:

- Given a divisor function, what partition function does it correspond to (if any)?
- Is it easy to know which one?
- What type of results come out from such a correspondence?
- Can this correspondence be used in the “other” theory?

Definition: series-divisor

The series divisor

$$Q(q) = \sum_{k=0}^{\infty} P(n) q^n$$

$$\sum_{k=1}^{\infty} kP(k)q^k = \sum_{k=1}^{\infty} \sigma^Q(k) q^k \sum_{k=0}^{\infty} P(k) q^k.$$

$$nP(n) = \sum_{k=1}^{n} \sigma^Q(k) P(n-k)$$

The coefficient of $q\frac{d}{dq}(Q(q)$

Idea # 1: Sum Lemma. The series divisor of a product of two series is the sum of their series-divisors. That is

$$\sigma^{AB}(k) = \sigma^A(k) + \sigma^B(k).$$

Calculus of series-divisors

$$\sum_{k=1}^{\infty} kP(k)q^k = \sum_{k=1}^{\infty} \sigma^Q(k)q^k \sum_{k=0}^{\infty} P(k)q^k.$$

$$1-q \rightsquigarrow (-1,-1,-1,-1,-1,-1,\dots)$$

$$\frac{1}{1-q} \rightsquigarrow (1,1,1,1,1,1,1,\dots)$$

$$\frac{1}{1-q^2} \rightsquigarrow (0,2,0,2,0,2,0,\dots)$$

$$\frac{1}{1-q^3} \rightsquigarrow (0,0,3,0,0,3,0,\dots)$$

$$\frac{1}{1-q^4} \rightsquigarrow (0,0,0,4,0,0,0,\dots)$$

$$\frac{1}{1+q} \rightsquigarrow (1,-1,1,-1,1,-1,1,\dots)$$

Calculus (cont.)

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} \rightsquigarrow \begin{aligned} &(1,1,1,1,1,1,1,1,1,1,\dots) \\ &+(0,2,0,2,0,2,0,2,0,2,\dots) \\ &+(0,0,3,0,0,3,0,0,3,0,\dots) \\ &+(0,0,0,4,0,0,0,4,0,0,\dots) \\ &=(1,1+2,1+3,1+2+4,\dots) \end{aligned}$$

$$\rightsquigarrow (\sigma(1),\sigma(2),\sigma(3),\sigma(4),\dots)$$

Thus the name: series-divisor

Ramanujan

$$\sum_{d=1}^{n} \sigma(d)p(n-d) = np(n)$$

A1: The correspondence

$$d_1(n) = \sigma(n) \leftrightsquigarrow \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \sum_{k=0}^{\infty} p(k) q^k$$

$$d_2(n) = \sigma(n) - 2\sigma(n/2) \leftrightsquigarrow \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} = \sum_{k=0}^{\infty} p_o(k) q^k$$

Partitions with all parts odd

$$d_3(n) = 2d_5(n) = 2\sigma(n/2) \leftrightsquigarrow \prod_{k=1}^{\infty} \frac{1}{1-q^{2k}} = \sum_{k=0}^{\infty} p_e(k) q^k$$

Partitions with all parts even

$$2d_4(n) = 2\big(\sigma(n) - \sigma(n/2)\big) \leftrightsquigarrow \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k} = \sum_{k=0}^{\infty} \overline{p}(k) q^k$$

Over partitions

$$(-1)^{n+1} 2d_4(n) \leftrightsquigarrow \prod_{k=1}^{\infty} (1+q^{2k-1})^2 (1-q^{2k}) = \varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}$$

Squares

$$d_6(n) = \sigma(n) - 4\sigma(n/2) = \leftrightsquigarrow \prod_{k=1}^{\infty} \frac{(1-q^{2k})}{(1-q^{2k-1})} = \psi(q) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}$$

Triangular numbers

$$d_7(n) = \sigma(n) - 2\sigma(n/2) = \leftrightsquigarrow \prod_{k=1}^{\infty} (1+q^k) = \sum_{k=0}^{\infty} p_d(k) q^k$$

Partitions with distinct parts

A2: Overpartitions

$$n\overline{p}(n) = 2\sum_{i=1}^{n} d_4(i)\overline{p}(n-i) = 2\sum_{i=1}^{n} \left(\sigma(i) - \sigma(i/2)\right)\overline{p}(n-i)$$

Proof

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n-1})} \rightsquigarrow d_1(n)+d_2(n) = 2d_4(n)$$

$$d_1(n) = \sum_{d|n} d$$

$$d_2(n) = \sum_{\substack{d|n \\ n \text{ odd}}} d$$

$$d_4(n) = \sum_{\substack{d|n \\ n/d \text{ odd}}} d = \sigma(n) - \sigma(n/2)$$

A3: Analogues of Ramanujan's recurrence

$$np_e(n) = \sum_{i=1}^{n} d_3(i) p_e(n-i) = 2\sum_{i=1}^{n} \sigma(i/2) p_e(n-i)$$

$$np_d(n) = \sum_{i=1}^{n} d_2(i) p_o(n-i) = \sum_{i=1}^{n} d_7(i) p_d(n-i)$$

$$= \sum_{i=1}^{n} \big(\sigma(i) - 2\sigma(i/2)\big) p_d(n-i)$$

$$n\overline{p}(n) = 2\sum_{i=1}^{n} d_4(i)\overline{p}(n-i) = 2\sum_{i=1}^{n} \big(\sigma(i) - \sigma(i/2)\big)\overline{p}(n-i)$$

Some more

$$nr_m(n) = m\sum_{j=1}^{n} 2(-1)^{j+1}\big(\sigma(j) - \sigma(j/2)\big)r_m(n-j)$$

$$nt_m(n) = m\sum_{j=1}^{n} \big(\sigma(j) - 4\sigma(j/2)\big)t_m(n-j)$$

$r_m(n) :=$ the number of ways n can be written as an ordered sum of m squares

$t_m(n) :=$ the number of ways n can be written as an ordered sum of m triangular numbers

Special cases

$r_1(0) = r_4(0) = r_8(0) = 1$

$$r_1(n) = \begin{cases} 2, & \text{if } n = j^2 \text{ for some } j > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$r_4(n) = 8\big(\sigma(n) - 4\sigma(n/4)\big)$$

$$r_8(n) = (-1)^{n+1}16\big(\sigma_3(n) - 16\sigma_3(n/2)\big) = (-1)^{n+1}16\widetilde{\sigma}_3(n)$$

$t_1(0) = t_4(0) = t_8(0) = 1$

$$t_1(n) = \begin{cases} 1, & \text{if } n = j(j+1)/2 \text{ for some } j > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$t_4(n) = \sigma(2n+1)$$

$$t_8(n) = \sigma_3(n+1) - \sigma_3((n+1)/2) = \overline{\sigma}_3(n+1)$$

$$8\sum_{j=1}^{n} d_6(j)\overline{\sigma}_3(n+1-j) = n\overline{\sigma}_3(n+1)$$

Idea#2: Power recursion

$$Q(q)^r = \sum_{n=0}^{\infty} P_r(n) q^n$$

$$\sum_{j=0}^{n} (n - (r/s+1)j)\, P_r(n-j) P_s(j) = 0$$

Gould (1974): s=1.
General case can be obtained from there.
Actually, r and s can be anything

$$nP_r(n) = \sum_{j=1}^{\infty} (-1)^{j+1} (2j+1)\,(n + (r/3-1)j(j+1)/2)\, P_r\big(n - j(j+1)/2\big)$$

Lehmer (1951): used a special case

Proof of the power recursion

$$rQ(q)^r \sum_{n=1}^{\infty} \sigma^Q(n) q^n = \sum_{n=0}^{\infty} n P_r(n) q^n$$

$$sQ(q)^s \sum_{n=1}^{\infty} \sigma^Q(n) q^n = \sum_{n=0}^{\infty} n P_s(n) q^n$$

Eliminate the generating function of series divisors

$$\frac{r}{s}\left(\sum_{n=0}^{\infty} P_r(n) q^n\right)\left(\sum_{n=0}^{\infty} n P_s(n) q^n\right) = \left(\sum_{n=0}^{\infty} n P_r(n) q^n\right)\left(\sum_{n=0}^{\infty} P_s(n) q^n\right)$$

Compare coefficients of q^n on both sides

$$\sum_{j=0}^{n} (n - (r/s+1)j)\, P_r(n-j) P_s(j) = 0$$

A3 (cont.)

$$nr_m(n) = -2\sum_{k=1}^{\infty}(n-(m+1)k^2)r_m(n-k^2)$$

Williams (2011)/Venkov (1970)

$$nt_m(n) = -\sum_{k=1}^{\infty}(n-(m+1)k(k+1)/2)t_m(n-k(k+1)/2)$$

Bal-Bhatnagar (2022-online)

$$2n\sigma(2n+1) = \sum_{j=1}^{\infty}(5j(j+1)-2n)\sigma(2n+1-j(j+1))$$

$$\textit{If } 5 \nmid n, \textit{ then } \sum_{j=0}^{\infty}\sigma(2n+1-j(j+1)) \equiv 0 \pmod 5$$

Obtained as special cases

Example

$$\sum_{j=0}^{\infty} \sigma\big(2n+1-j(j+1)\big) \equiv 0 \pmod 5$$

sum of divisors

0	1	3	4	7	6	12	8	15	13
18	12	28	14	24	24	31	18	39	20
42	32	36	24	60	31	42	40	56	30
72	32	63	48	54	48	91	38	60	56
90	42	96	44	84	78	72	48	124	57
93	72	98	54	120	72	120	80	90	60

$$n = 1: \quad \sigma(3) + \sigma(3-2) = 4 + 5 = 5$$

$$n = 2: \quad \sigma(5) + \sigma(5-2) + \sigma(5-6) = 6 + 4 + 0 = 10$$

$$n = 3: \quad \sigma(7) + \sigma(7-2) + \sigma(7-6) = 8 + 6 + 1 = 15$$

$$n = 4: \quad \sigma(9) + \sigma(7) + \sigma(3) = 13 + 8 + 4 = 25$$

Two more for overpartitions

$$\sigma(n) = \sum_{i=-\infty}^{\infty} (-1)^i \left(n - \frac{i(3i-1)}{2}\right) p\left(n - \frac{i(3i-1)}{2}\right)$$

Euler

$$\sum_{i=1}^{n} \sigma(n-i)\sigma(i) = \sum_{i=-\infty}^{\infty} (-1)^{i+1} \left(\frac{i(3i-1)}{2}\right)\left(n - \frac{i(3i-1)}{2}\right) p\left(n - \frac{i(3i-1)}{2}\right)$$

Glaisher

$$2d_4(n) = 2\big(\sigma(n) - \sigma(n/2)\big) = n\overline{p}(n) + 2\sum_{i=1}^{\infty}(-1)^i (n-i^2)\overline{p}(n-i^2)$$

$$2\sum_{i=1}^{n} d_4(n-i)d_4(i) = \sum_{i=1}^{\infty}(-1)^{i+1} i^2 (n-i^2)\overline{p}(n-i^2)$$

A4

Proof of Ramanujan's recurrences

$$p(5m+4) \equiv 0 \pmod 5$$

$$\tau(5n+5) \equiv 0 \pmod 5$$

$$\prod_{k=1}^{\infty}(1-q^k)^{24} = \sum_{n=0}^{\infty}\tau(n+1)q^n$$

Jacobi

$$\prod_{k=1}^{\infty}(1-q^k)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}$$

Notation $$Q(q)^r = \sum_{n=0}^{\infty} P_r(n) q^n$$

$$Q(q) = \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

$r = 1 :$ $\quad P_1(n) = p(n) = \text{partitions}$

$r = -24 :$ $\quad P_{-24}(n) = \tau(n+1) = \text{Ramanujan's } \tau \text{ function}$

$r = -3 :$ $\quad P_{-3}(n) = \begin{cases} (-1)^k(2k+1) & \text{if } n = \frac{k(k+1)}{2} \\ 0 & \text{otherwise} \end{cases}$

The proof!

$P_r(5m+4) \equiv 0 \pmod 5,\ \textit{if } r \equiv 1 \pmod 5$

$$nP_r(n) = \sum_{j=1}^{\infty}(-1)^{j+1}(2j+1)\,(n+(r/3-1)j(j+1)/2)\,P_r\big(n-j(j+1)/2\big)$$

$$m=0:\ \ 4P_r(4) = (9+r)P_r(3) - 5(r+1)P_r(2) \text{ so } P_r(4) \equiv 0 \pmod 5$$

$$(-1)^{j+1}(2j+1)\,(n+(r/3-1)j(j+1)/2)\,P_r\big(n-j(j+1)/2\big)$$

$$j \equiv 4,5 \pmod 5: \qquad \big({*}{*}{*}\big)P_r(5m+4-5k) \equiv 0 \pmod 5$$

$$j \equiv 3 \pmod 5: \qquad (-1)^{j+1}2(2r-2)P_r(**) \equiv 0 \pmod 5$$

$$j \equiv 2 \pmod 5: \qquad (-1)^{j+1}5(1+r)P_r(**) \equiv 0 \pmod 5$$

$$j \equiv 1 \pmod 5: \qquad (-1)^{j+1}5(1+r)P_r(**) \equiv 0 \pmod 5$$

’

(1) $P_r(5m+1) \equiv 0 \pmod 5$, if $r \equiv 0 \pmod 5$
(2) $P_r(5m+2) \equiv 0 \pmod 5$, if $r \equiv 2 \pmod 5$
(3) $P_r(5m+3) \equiv 0 \pmod 5$, if $r \equiv 4 \pmod 5$
(4) $P_r(5m+4) \equiv 0 \pmod 5$, if $r \equiv 1 \pmod 5$

Some cases of Chan and Wang (2019) covered (r can be rational)

Let $r \equiv 0 \pmod 3$. Then

$$P_r(3m+k) \equiv 0 \pmod 3 \text{ for } k = 1, 2.$$

Thank you

Hartosh Singh Bal (Caravan) and Gaurav Bhatnagar (Ashoka)

July 7, 2022

Ramanujan's entry

Let $f(q) = \sum_{k=1}^{\infty} \frac{A_k q^k}{k}$ and $e^{f(q)} = \sum_{n=0}^{\infty} P_n q^n$

Then

$$P_0 = 1 \qquad nP_n = \sum_{k=1}^{n} A_k P_{n-k}$$

Berndt: Part II

$$\sum_{d=1}^{n} \sigma(d)p(n-d) = np(n)$$

Berndt: Part IV