The Worpitzky identity in Coxeter groups of type D

$$(x+1)^n = \sum_{k=0}^{n-1} \binom{x+n-k}{n} A_{n,k}$$

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The Descent set

Definition

Let $\pi \in S_n$.

$$\mathrm{Des}(\pi) := \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}.$$

$$\mathrm{des}(\pi) := |\,\mathrm{Des}(\pi)|.$$

Example

Let $\pi = [42315]$. Then: $\mathrm{Des}(\pi) = \{1, 3\}$ and $\mathrm{des}(\pi) = 2$.

Eulerian numbers and Worpitzki's identity for type A

The Eulerian number $A_{n,k}$ counts the number of permutations in S_n having k descents:

Definition

$$A_{n,k} = |\{\pi \in S_n : \mathrm{des}(\pi) = k\}|.$$

Theorem (Worpitzky)

For all $m, n \in \mathbb{N}$:

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The group B_n

Definition

A **signed permutation** is a permutation π on the set $\{\pm 1, \ldots, \pm n\}$ with the property that $\pi(-i) = -\pi(i)$ for all i.

It suffices to specify $\pi(i)$ for $i > 0$, so we can think of a signed permutation as a permutation with the additional property that some of the entries can be negative.

Example

$$\pi = [2, -4, 3, -1, 7, -5, 6] \in B_7$$

For $\pi \in B_n$: $\quad \mathrm{neg}(\pi) = |\{i : \pi(i) < 0, 1 \leq i \leq n\}|$.

Example

Let $\pi = [-1, 2, -5, 4, 3]$. Then $\mathrm{neg}(\pi) = 2$.

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The group D_n

Definition

The group of signed permutations has an index 2 subgroup consisting of signed permutations with an even number of negative entries.

$$D_n = \{\pi \in B_n \mid neg(\pi) \equiv 0 \pmod 2\}$$

Example

Following Petersen, we write the elements of D_n as 'forked permutations'. For example, the even-signed permutation $w = [3, 2, -4, -1] \in D_4$ (in the usual window notation) will be written here as

$$w = \left[1, 4, -2, \begin{matrix} 3 \\ -3 \end{matrix}, 2, -4, -1\right].$$

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Eulerian numbers of type D

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For $\pi \in B_n$, define: $\mathrm{Des}_A(\pi) = \{i : \pi(i) > \pi(i+1), 1 \leq i \leq n-1\}$.
For $\pi \in D_n$, define:

$$\mathrm{Des}_D(\pi) = \begin{cases} \mathrm{Des}_A(\pi) \cup \{-1\} & \pi(1)+\pi(2) < 0 \\ \mathrm{Des}_A(\pi) & \pi(1)+\pi(2) > 0 \end{cases}$$

and denote: $\mathrm{des}_D(\pi) = |\mathrm{Des}_D(\pi)|$.

$$D_{n,k} = |\{\pi \in D_n : \mathrm{des}_D(\pi) = k\}|$$

are the Eulerian numbers of type D.

Example

$$\pi = \left[-4,-1,5,-6,-2 \;\substack{-3\\3}\; ,2,6,-5,1,4\right].$$

Then $\mathrm{Des}_D(\pi) = \{-1,3\}$.

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Eulerian numbers for type D - the q-analogue

Definition

For the q-analogue: let

$$D_{n,k}(q) = \sum_{\pi \in D_n : \mathrm{des}_D(\pi) = k} q^{\mathrm{neg}_2(\pi)},$$

where

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The Worpitzki identity for type D

Theorem (Brenti 94')

$$(1+2m)^n - 2^{n-1}(n(1^{n-1}+\cdots+m^{n-1})) = \sum_{k=0}^{n} \binom{n+m-k}{n} D_{n,k}.$$

The q-analogue:

$$(1+2m)((1+q)m)^{n-1} - (1+q)^{n-1} n \sum_{i=1}^{m} i^{n-1} = \sum_{k=0}^{n} \binom{n+m-k}{n} D_{n,k}(q)$$

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A bit of history

- **Foata and Schützenberger (1970), Rawlings (1981):** A proof of the Worpitzky identity for the Coxeter group of type A in a combinatorial way.
- **Brenti (1994):** Generalizations of the Worpitzky identity (in their q-versions) for Coxeter groups of types B and D, using the algebraic Coxeter definition of the descents in these groups.
- **Borowiec and Młotkowski (2016):** Generalization of Worpitzky's identity to Coxeter groups of types B and D, using a different set of Eulerian numbers.

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P-partitions

- Let $P = \{p_1, \ldots, p_n\}$ be a partially ordered set (poset), labeled by the set $[n] = \{1, \ldots, n\}$, with the partial order $<_P$.
- We identify each element in P with its label.
- A *P-partition (of type A)* is an order-preserving map $f : [n] \to \mathbb{Z}$ satisfying:
  1. $f(i) \leq f(j)$, if $i <_P j$,
  2. $f(i) < f(j)$, if $i <_P j$ and $j > i$ in $\mathbb{Z}$.

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The D_n-poset

Definition (Stembridge, 2008)

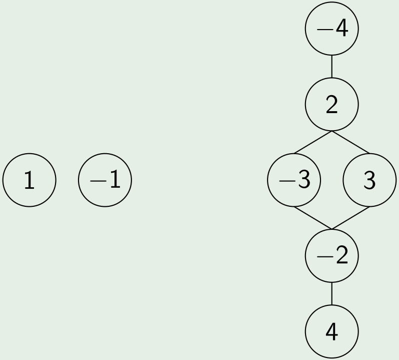
A *D_n-poset* is the set $P = \{\pm 1, \pm 2, \ldots, \pm n\}$ with a partial order $<_P$, satisfying the following conditions:

1. If $i <_P j$, then $-j <_P -i$,
2. If $-i <_P i$, then there is some $j \neq \pm i$ such that $-i <_P j <_P i$ (*'fork' condition*).

The second condition means that each (Hasse diagram of a) D_n-poset must have a "fork" in the middle.

An example of a D_4-poset

Example



P-partitions of type D

Definition

Let P be a D_n-poset and let $(X, \preccurlyeq)$ be a countable totally ordered set. A *P-partition P of type D* is an order-preserving map $f : [\pm n] \to X$ satisfying for all i, j:

- $f(i) \preccurlyeq f(j)$, if $i <_P j$,
- $f(i) \prec f(j)$, if $i <_P j$ and $i > j$ in $\mathbb{Z}$,
- $f(-i) = -f(i)$.

Here we use $(X, \preccurlyeq) = (\mathbb{Z}, \preccurlyeq)$ (with the convention that when $x \preccurlyeq y$ and $x \neq y$, we write $x \prec y$), where the order relation is defined as:

$$\mathbf{0} \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots .$$

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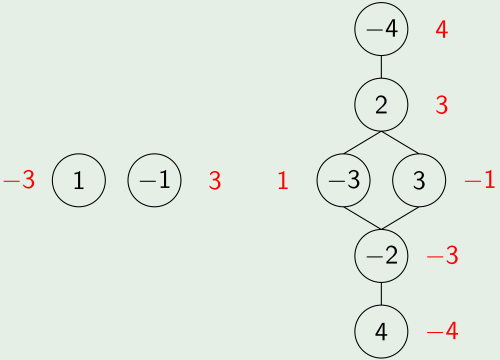


The set of P-partitions and a general example

Definition

Let $\mathcal{A}(P)$ denote the set of all P-partitions of type D of the poset P.

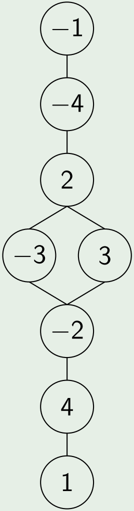
Example

Every $\pi \in D_n$ induces a D_n-poset by defining $\pi(i) <_\pi \pi(i+1)$ for $1 \leq i \leq n-1$.

Example

$$\pi = [1, 4, -2, \begin{matrix} 3 \\ -3 \end{matrix}, 2, -4, -1]$$

induces the D_4-poset:



Linear extensions

A **linear extension** of a D_n- poset P is a D_n-poset which is identical to P as a set but maximally refines the partial order.

Example

$$\left\{\begin{array}{l}
[1,4,-2,\begin{smallmatrix}3\\-3\end{smallmatrix},2,-4,-1],\\
[-1,4,-2,\begin{smallmatrix}-3\\3\end{smallmatrix},2,-4,1],\\
[4,1,-2,\begin{smallmatrix}3\\-3\end{smallmatrix},2,-1,-4],\\
[4,-1,-2,\begin{smallmatrix}-3\\+3\end{smallmatrix},2,1,-4],\\
[4,-2,1,\begin{smallmatrix}3\\-3\end{smallmatrix},-1,2,-4],\\
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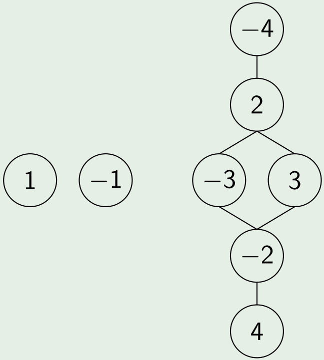

The fundamental theorem of P-partitions

Theorem

Let P be a D_n-poset.
Then

$$\mathcal{A}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi),$$

where $\mathcal{L}(P)$ is the set of linear extensions of P.

We deal with the anti-chain D_n-poset:

$$P = \{\pm 1, \pm 2, \dots, \pm n\}$$

with no relations at all. In this case, the set of linear extensions of P coincides with the group D_n.

How to associate a P-partition with a D_n-permutation

- In order to associate an element $\pi \in D_n$ to a P-partition f, we read the elements of f in the order $\preccurlyeq$ and record the location of each element.
- Identical positive (negative) elements are read from left (right) to left (right) respectively.
- We record a negative location if the element is negative.
- Whenever we get a permutation $\pi \in B_n - D_n$ we switch the sign of $\pi(1)$.

Example

The P - partition

$$f = (f(1), f(2), \ldots, f(7)) = (-2, 1, 3, 2, -3, 2, -3)$$

is associated with the permutation

$$\pi = \left[-3, 5, 7, -6, -4, 1, \begin{matrix} -2 \\ 2 \end{matrix}, -1, 4, 6, -7, -5, 3 \right] \in D_7.$$

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How to construct $\mathcal{A}(\pi)$

- In the other direction, we show now how to find the set $\mathcal{A}(\pi)$ for a given $\pi \in D_n$.
- Each P-partition $f \in \mathcal{A}(\pi)$ must satisfy the following inequalities:

$$f(\pi(1)) \preccurlyeq f(\pi(2)) \preccurlyeq \cdots \preccurlyeq f(\pi(n))$$

and $f(\pi(-1)) \preccurlyeq f(\pi(2))$, where for $1 \leq i \leq n-1$, the condition $i \in \mathrm{Des}_D(\pi)$ forces the strict inequality:

$$f(\pi(i)) \prec f(\pi(i+1)),$$

while the condition $-1 \in \mathrm{Des}_D(\pi)$ forces the strict inequality:

$$f(\pi(-1)) \prec f(\pi(2)).$$

Example

For $\pi = \left[4, -2, -1, \begin{matrix} -3 \\ 3 \end{matrix}, 1, 2, -4\right] \in D_4$, the elements $f \in \mathcal{A}(\pi)$ must satisfy:

- $$f(-3) \preccurlyeq f(1) \preccurlyeq f(2) \prec f(-4)$$
  as well as $f(3) \prec f(1)$, together with the sign conditions:
- $f(1) > 0, f(2) > 0, f(4) < 0$.

Since there is no restriction on the sign of $f(3)$ (in order to contain functions with odd number of negative values), we have three possibilities for the values of f:

1. $0 < f(3) < |f(1)| \leq |f(2)| < |f(4)|$.
2. $0 < f(-3) < |f(1)| \leq |f(2)| < |f(4)|$.
3. $0 = f(3) < |f(1)| \leq |f(2)| < |f(4)|$.

Example

For $\pi = \left[4, -2, -1, \begin{matrix} -3 \\ 3 \end{matrix}, 1, 2, -4\right] \in D_4$, the elements $f \in \mathcal{A}(\pi)$ must satisfy:

- $$f(-3) \preccurlyeq f(1) \preccurlyeq f(2) \prec f(-4)$$

  as well as $f(3) \prec f(1)$, together with the sign conditions:
- $f(1) > 0, f(2) > 0, f(4) < 0$.

Since there is no restriction on the sign of $f(3)$ (in order to contain functions with odd number of negative values), we have three possibilities for the values of f:

1. $0 < f(3) < |f(1)| \leq |f(2)| < |f(4)|$.
2. $0 < f(-3) < |f(1)| \leq |f(2)| < |f(4)|$.
3. $0 = f(3) < |f(1)| \leq |f(2)| < |f(4)|$.

Proof idea:

Let $\mathcal{A}_m(P) = \{f \in \mathcal{A}(P) \mid \forall i, f(i) \le m\}$.
Consider the anti-chain D_n-poset $P = \{0, \pm 1, \ldots, \pm n\}$. By the fundamental theorem of P-partitions of type D we have

$$(2m+1)^n = |\mathcal{A}_m(P)| = \sum_{\pi \in D_n} |\mathcal{A}_m(\pi)| =$$

$$= \sum_{\pi \in D_n} \binom{m + n - \mathrm{des}_D(\pi)}{n} +$$

$$+ \sum_{\pi \in D_n} \binom{m + n - \mathrm{des}_D(\pi) - 1 + |\mathrm{Des}_D(\pi) \cap \{-1, 1\}|}{n} =$$

$$= \sum_{\pi \in D_n} \binom{m + n - \mathrm{des}_D(\pi)}{n} + 2^{n-1} n (1^n + \cdots + m^n)$$

$\square$

Lemma

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$$\sum_{\pi \in D_n} \binom{m+n-\mathrm{des}_D(\pi)-1+|\mathrm{Des}_D(\pi)\cap\{-1,1\}|}{n} = 2^{n-1} n(1^n + \cdots + m^n)$$

Proof.

The R.H.S. counts the number of elements in the set of vectors of length n over the alphabet $\Sigma = \{1, \ldots, m+1\}$ such that each entry can be either positive or negative, the number of negative entries is even, and the smallest entry in absolute value appears exactly once.

The L.H.S counts the same set of vectors by ordering them according to their associated permutation in D_n. □

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The 'ignore 1' descent set for D_n

Definition

Let $\pi \in D_n$. Define

$$\mathrm{Des}_{D,2}(\pi) = \{i \in [2, \ldots, n-1] \mid \pi(i) > \pi(i+1)\}$$

and let

$$\mathrm{des}_{D,2}(\pi) = |\mathrm{Des}_{D,2}(\pi)|.$$

Let

$$A_{D,2}(n,k) = |\{\pi \in D_n \mid \mathrm{des}_{D,2}(\pi) = k\}.$$

Another formulation of the lemma above

Theorem

Let $n, m \in \mathbb{N}$. Then:

$$2^{n-1}n(1^{n-1}+\cdots+m^{n-1}) = \sum_{k=1}^{n} A_{D,2}(n,k)\binom{n+m-k-1}{n}$$

An algebraic view

- Let H_n be the parabolic subgroup of D_n generated by the Coxeter generators $s_2, \dots, s_{n-1}$.
- H_n can be identified with $S_{n-1} = \langle s_2, \dots, s_{n-1} \rangle$, hence we have that

$$|D_n/H_n| = \frac{2^{n-1} n!}{(n-1)!} = 2^{n-1} n.$$

- Two elements $\sigma, \tau \in D_n$ share the same left coset if and only if $|\sigma(1)| = |\tau(1)|$ and

$$\{\sigma(i) \mid \sigma(i) < 0, \ i \geq 2\} = \{\tau(i) \mid \tau(i) < 0, \ i \geq 2\}.$$

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Hence,

- $$\begin{aligned}\mathrm{des}_{D,2}(\pi) &:= |\{i \mid \pi(i) > \pi(i+1),\ 2 \le i \le n-1\}| = \\ &= \mathrm{des}_D(\pi) - |\mathrm{Des}_D(\pi) \cap \{-1, 1\}|\end{aligned}$$

  ignores the descents in places -1 and 1.
- The sum
  $$\sum_{\pi \in \sigma H} \binom{m+n-\mathrm{des}_{D,2}(\pi)-1}{n}$$
  is constant on each coset σH.
- Recall:
  $$2^{n-1} n(1^{n-1} + \cdots + m^{n-1}) = \sum_{\pi \in D_n} \binom{n+m-\mathrm{des}_{D,2}(\pi)-1}{n}.$$

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A new Worpitzky-like identity in type A

This leads to the following Worpitzky-like identity on S_{n-1} :

Corollary

$$1^{n-1} + \cdots + m^{n-1} = \sum_{\pi \in S_{n-1}} \binom{n + m - \mathrm{des}_A(\pi) - 1}{n}.$$

Thank you for your attention!