Partition Identities for k-Regular Partitions with Distinct Parts

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The definition of a k-regular partition depends on whether you are doing group theory or partitions.
For group theory, k-regular means no part appears k or more times.
For partitions, k-regular means no part is divisible by k.
Of course, these two definitions are tied together in:

Glaisher's Theorem

The number of partitions of N in which no part is repeated k or more times equals the number of partitions of N into parts not divisible by k.

When discussing partitions with distinct parts we mean by k-regular that no part is divisible by k.

There is a general theorem on k-regular partitions with distinct parts. In this theorem the parts are required to be "close together" rather than separated.

Theorem (Olsson et al.)

The number of k-regular partitions of n into distinct parts equals the number of partitions of n in which only multiples of k may be repeated, the smallest part is $< k$, the difference between consecutive parts is $\leq k$, and $< k$ if either part is divisible by k.

For example, when $k = 4$ the eight 4-regular partitions of 12 with distinct parts are

$11+1, 10+2, 9+3, 9+2+1, 7+5, 7+3+2, 6+5+1, 6+3+2+1.$

The partitions satisfying the closeness conditions are

$$7+4+1, 7+3+2, 6+5+1, 6+4+2, 6+3+2+1,$$

$$5+4+3, 5+4+2+1, 4+4+3+1.$$

Note: $4+4+4$ is excluded because the smallest part is not < 4.

Krishna Alladi was the first to state the following general theorem on k-regular partitions with distinct parts.

Alladi's Theorem

The number of k-regular partitions of n with distinct parts equals the number of partitions of n into odd parts that are not divisible by k.

Figure: Krishnaswami Alladi [1]

[1] https://akuncu.files.wordpress.com/2019/10/alladi1.jpg

The cases $k = 1$ and $k = 2$ are tautologies.
$k = 3$ fits into Schur's 1926 Theorem.

Schur's Theorem

The number of partitions of n into distinct parts not divisible by 3 equals the number of partitions of n into parts $\equiv \pm 1 \pmod 6$ *equals the number of partitions of n in which parts differ by at least 3 and multiples of 3 differ by at least 6.*

Figure: Issai Schur [2]

[2]https://upload.wikimedia.org/wikipe...commons/1/1f/Schur.jpg

While the $k = 1$ and $k = 2$ cases of Alladi's Theorem are tautologies, both Alladi and Schur proved identities for 2-regular partitions with distinct parts.

Alladi's Theorem

The number of 2-regular partitions of n with distinct parts equals the number of partitions of n where

1. *2 is not a part*
2. *difference between parts is ≥ 6 with strict inequality if either part is even.*

For example, for $n = 20$, the seven 2-regular partitions with distinct parts are:

$19 + 1, 17 + 3, 15 + 5, 13 + 7, 11 + 9, 11 + 5 + 3 + 1, 9 + 7 + 3 + 1.$

The partitions of 20 satisfying (i) and (ii) are

$$20, 19 + 1, 17 + 3, 16 + 4, 15 + 5, 14 + 6, 13 + 7.$$

Schur's Theorem

The number of 2-regular partitions of n with distinct parts equals the number of partitions of n in which

1. *no part is congruent to* $2 \pmod 4$, *and*
2. *parts differ by* ≥ 4 *with strict inequality if either part is a multiple of 4.*

Here the second class of partitions at $n = 20$ is

$$20, 19 + 1, 17 + 3, 16 + 4, 15 + 5, 13 + 7, 13 + 6 + 1.$$

Now for Schur's $k = 3$ Theorem, Alladi added a fourth set of partitions to the three in Schur's Theorem:
The number of partitions of n into distinct parts not divisible by k (i.e. k-regular partitions with distinct parts) equals the number of partitions of n into odd parts none repeated more than $k - 1$ times.
For $N = 9$ we have the following:

Distinct, none $\equiv 0 \pmod 3$	$8+1$	$7+2$	$5+4$
$\equiv \pm 1 \pmod 6$	$7+1+1$	$5+\sum_{i=1}^{4} 1$	$\sum_{i=1}^{9} 1$
$a_i - a_{i+1} \geq 3, (\geq 6$ mults $3)$	9	$8+1$	$7+2$
odd parts, at most twice	9	$7+1+1$	$5+3+1$

There are numerous proofs and refinements of Schur's Theorem.

Some proofs are

INTRINSIC

and some are

EXTRINSIC.

Intrinsic proofs rely on factorization of polynomial generating functions.

Extrinsic proofs rely on some sort of q-hypergeometric identity.

To illustrate we begin with Gleissberg's refinement of Schur's Theorem:

Gleissberg's Theorem

Let $A(m,n)$ denote the number of partitions of n into m distinct parts not divisible by three. Let $B(m,n)$ denote the number of partitions of n into parts differing by 3 and by at least 6 between multiples of 3 m denotes the total number of parts plus the number divisible by 3 (i.e. multiples of 3 are counted twice). Then

$$A(m,n) = B(m,n).$$

Let $\beta_N(m,n)$ denote the number of partitions counted by $B(m,n)$ where the largest part is $\leq N$.
Define

$$d_N(x;q) = d_N(x) := \sum_{m,n\geq 0} \beta_N(m,n)x^m q^n.$$

Gleissberg's proof of his theorem avoided generating functions.

The generating function version of Gleissberg's Theorem is:

$$d_{3N}(x) = (1+xq)(1+xq^2)d_{3N-4}(xq^3).$$

Note: this implies

$$d_\infty(x) = (1+xq)(q+xq^2)d_\infty(xq^3)$$

and iteration yields

$$d_\infty(x) = \prod_{\substack{n=1 \\ 3\nmid n}}^{\infty} (1+xq^n).$$

Gleissberg's Theorem is an example of an intrinsic proof of Schur's Theorem.

Intrinsic proofs all rely on recurrences. Here is a sketch of the polynomial version of Gleissberg's proof:
First

$$
\begin{aligned}
d_{3N-1}(x) &= d_{3N-2}(x) + xq^{3N-1} d_{3N-4} \\
d_{3N-2}(x) &= d_{3N-3}(x) + xq^{3N-2} d_{3N-5} \\
d_{3N-3}(x) &= d_{3N-4}(x) + xq^{3N-3} d_{3N-7}.
\end{aligned}
$$

Second, one eliminates instances of d_{3j-2} and d_{3j-3} to obtain

$$d_{3N-1}(x) = (1 + xq^{3N-1} + xq^{3N-2})d_{3N-4}(x) \\ + x^2q^{3N-3}(1 - q^{3N-3})d_{3N-7}$$

Third, replace x by xq^3 in the original recurrences. Then eliminate instances of d_{3j-1} and d_{3j-2} revealing that $d_{3N+3}(x)$ and $d_{3N-1}(xq^3)$ satisfy the same second-order recurrence.
Finally one shows by computation that for $N = 1$ and $N = 2$,

$$d_{3N}(x) = (1 + xq)(1 + xq^2)d_{3N-4}(xq^3).$$

Here is an extrinsic proof of Schur's Theorem:
Recall

$$d_{3N-1}(x) = (1 + xq^{3N-1} + xq^{3N-2})d_{3N-4} + x^2 q^{3N-3}(1 - q^{3N-3})d_{3N-7}.$$

Set $\overline{d}_N = d_{3N-1}/(q^3;q^3)_N$.

Hence

$$(1q^{3N})\overline{d}_N = (1 + xq^{3N-1} + xq^{3N-2})\overline{d}_{N-1} + x^2 q^{3N-3}(1 - q^{3N-3})d_{N-2}.$$

Let $F(t) = \sum_{N \geq 0} \overline{d}_N t^N$.

Multiply the last equation by t^N and sum over $N \geq 0$:

$$F(t) - F(tq^3) = tF(t) + t(xq^2 + xq)F(tq^3) + t^2x^2q^3F(tq^3).$$

So

$$(1-t)F(t) = (1 + txq + txq^2 + t^2x^2q^3)F(tq^3)$$

Iterating yields

$$F(t) = \prod_{N=0}^{\infty} \frac{(1 + xtq^{3N+1})(1 + xq^{3N+2})}{(1 - tq^{3N})}.$$

Or

$$(1-t)\sum_{N\geq 0}\frac{d_{3N-1}t^N}{(q^3;q^3)_N}=\frac{(-xtq;q^3)_\infty(-xtq^2;q^3)_\infty}{(tq^3;q^3)_\infty}.$$

Letting $t\to 1^-$ and applying Abel's Lemma,

$$d_\infty=(-xq;q^3)_\infty(-xq^2;q^3)_\infty.$$

Now their is a Gleissbergesque theorem for Alladi's addition to Schur's Theorem:

Theorem

Let $C(m, n)$ denote the number of partitions of n into m odd parts none repeated more than twice. Let $D(m, n)$ denote the number of partitions of n into parts that differ by at least 3 (and at least 6 between multiples of 3) where m counts the number of parts plus the number of even parts (i.e. each even part is counted twice). Then

$$C(m, n) = D(m, n).$$

Just as with Gleissberg, there is an intrinsic proof here as well.
Let $\Delta_N(m,n)$ denote the number of partitions counted by $D(m,n)$ where the largest part is $\leq N$. Define

$$\delta_N(x;q) = \delta_N(x) := \sum_{m,n\geq 0} \Delta_N(m,n) x^m q^n.$$

Now

$$\delta_{6N+2}(x) := (1 + xq + x^2q^2)\delta_{6N-1}(xq^2).$$

Let $N \to \infty$ and iterate

$$\delta_\infty(x) := \prod_{n=0}^{\infty} (1 + xq^{2n+1} + x^2q^{4n+2}).$$

Indeed, much more is true:

$$d_{6N-1}(x) = \prod_{j=1}^{N}(1 + xq^{2j-1} + x^2q^{4j-2})\overbrace{(1 + O(q^{2N}))}^{\text{polynomial}}.$$

Not only is this an "intrinsic-er" proof, it also reveals that the Alladi contribution to Schur's Theorem is more intrinsic than the original formulation by Schur.

I should note that this is only a small part of the story of 3-regular partitions with distinct parts.
I have told only that part of the story which illustrates the nature of intrinsic and extrinsic proofs.

Actually, prior to much of the discovery of the material just presented on $k = 3$, the 5-regular partitions with distinct parts arose in a problem in group theory considered by Jorn Olsson and Christine Bessenrodt.

In the 1974 A. M. S. memoir *The General Rogers-Ramanujan Theorem*, we find

Conjecture 2 (slightly altered)

The number of 5-regular partitions of N with distinct parts equals the number of N of the form

$$N = \sum_{i=1}^{\infty} f_i \cdot i \ (f_i \text{ is the frequency of } i)$$

where

a. If $5 \nmid i$, then $f_i \leq 1$.
b. $f_i + f_{i+1} + \cdots + f_{i+4} \leq 2$.
c. $f_{5i} + f_{5i+1} + \cdots + f_{5i+4} \leq 2$.
d. $f_{5i+2} + f_{5i+3} \leq 1$.
e. $f_{5i+4} + f_{5i+6} \leq 1$.

Thus if $n = 10$ the 5-regular partitions with distinct parts are

$$9 + 1, 8 + 2, 7 + 3, 6 + 4, 6 + 3 + 1, 4 + 3 + 2 + 1.$$

The partitions satisfying (a)-(e) are

$$10, 9 + 1, 8 + 2, 7 + 3, 6 + 3 + 1, 5 + 5.$$

Note that $6 + 4$ is out by (e) and $4 + 3 + 2 + 1$ is out by (d).

I remark that in Schur's Theorem the difference condition part may be stated

$$f_i + f_{i+1} + f_{i+2} \leq 1$$

and

$$f_{3i} + f_{3i+1} + f_{3i+2} + f_{3i+3} \leq 1.$$

Now there are 15 possible partitions that satisfy (a)-(e) and have parts in $[5j+1, 5j+5]$. These are ordered lexicographically and then 15 polynomial generating functions $S_N(j,x)$ are defined where the "largest" part(s) equal j and all parts satisfy (a)-(e) and are $\leq 5N+5$.
THEN

$$S_N(15,x) = (1+xq)(1+xq^2)(1+xq^3)(1+xq^4)S_{N-1}(9,xq^5).$$

After this lengthy introduction, we now arrive at something which (I hope) is new. The obvious question is: what about 4-regular partitions into distinct parts?

In the case of 4-regular partitions into distinct parts, the only result I could find involved overpartitions.

Overpartitions are partitions in which the first occurrence of a number may be overlined. Thus the eight overpartitions of 3 are:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

Theorem

The number of 4-regular partitions of n into distinct parts equals the number of overpartitions of n subject to the following conditions

a. *Only odd parts* ≥ 3 *may be overlined.*
b. *The difference between any two parts is* ≥ 4 *and at* > 4 *if one of the parts is divisible by 4.*
c. *Also, the difference is* > 4 *if the larger part is overlined.*

For example, the six 4-regular partitions of 10 into distinct parts are $10, 9+1, 7+3, 7+2+1, 6+3+1, 5+3+2$. The overpartitions satisfying (a), (b), and (c) are

$$10, 9+1, \overline{9}+1, 8+2, 7+3, 7+\overline{3}.$$

Note: $\overline{7}+3$ is excluded by (c).

Given all that has gone before, we would expect a Gleissbergesque refinement.

Theorem

Let $A(m, n)$ denote the number of 4-regular partitions of n into distinct parts. Let $B(m, n)$ denote the number of overparitions of n subject to the conditions (a), (b), and (c), where m denotes the number of parts plus the number of overlined parts plus the number of parts divisible by 4.
Then

$$A(m, n) = B(m, n).$$

We shall give an extrinsic proof of this theorem because there is a 60+ -year-old Lemma buried in the literature that nails it and makes life easy.

Lemma

Let λ_n be defined by

$$\sum_{n\geq 0} \frac{\lambda_n x^n}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{(1 + a_1 x q^n + a_2 x^2 q^{2n} + a_3 x^3 q^{3n})}{(1 - x q^{n-1})}$$

and let D_n be defined by

$$\begin{aligned} D_{-1} = 0, D_0 &= 1, D_1 = 1 + a_1 q, \\ D_2 &= 1 + a_1 q + a_1 q^2 + a_1^2 q^3 + a_2 q^2 - a_2 q^3 + a_3 q^3, \\ \text{if } n > 2, D_n &= (1 + a_1 q^n) D_{n-1} + a_2 q^n (1 - q^{n-1}) D_{n-2} \\ &\quad + a_3 q^{2n-1} (a_1 + q^{n-1}) D_{n-3} + a_3^2 q^{3n-3} D_{n-4}. \end{aligned}$$

$$\text{Then } D_n = \sum_{0\leq 2s\leq n} \begin{bmatrix} n-s \\ s \end{bmatrix} a_3^s q^{(n+1)s - \binom{s}{2}} \lambda_{n-2s}$$

where $\begin{bmatrix} A \\ B \end{bmatrix}$ *is the q-binomial coefficient.*

$$q \to q^6, a_1 = q^{-1} + q^{-2} + q^{-4},$$
$$a_2 = q^{-3} + q^{-5} + q^{-6}, a_3 = q^{-7}$$

yields via Abel's Lemma again

The "Big" Göllnitz Theorem

The number of partitions of n into distinct parts congruent to $2, 4,$ *or* $5 \pmod 6$ *equals the number of partitions of n where*

a. *Neither 1 nor 3 are parts.*
b. *Parts differ by at least 6 and by more than 6 if one of the parts is congruent to* $0, 1,$ *or* $3 \pmod 6$.

For our theorem on 4-regular partitions with distinct parts,

$$\begin{aligned} q &\to q^4 \\ a_1 &= t(q^{-1} + q^{-2} + q^{-3}), \\ a_2 &= t^2(q^{-3} + q^{-4} + q^{-5}), \\ a_3 &= t^3 q^{-6} \end{aligned}$$

In this case the numbers have the following order:

$$1 < 2 < 3 < \overline{3} < 4 < 5 < \overline{5} < 6 < 7 < \overline{7} < \cdots$$

In this case,

$$D_{4n-1} = \sum_{0 \leq 2s \leq n} \begin{bmatrix} n-s \\ s \end{bmatrix}_{q^4} t^{3s} q^{4ns-2s^2} \lambda_{n-2s}$$

Of course, Alladi's Theorem tells us that the number of 4-regular partitions of n into distinct parts also equals the number of partitions of n into odd parts none repeated more than thrice.

Theorem

Let $\alpha(m, n)$ denote the number of partitions of n into m odd parts none appearing more than thrice. Let $\beta(m, n)$ denote the number of overpartitions of n subject to the previous conditions (a)-(c), where now m is a weighted count of the parts:
odd overlined parts are counted with weight 3
even parts are counted with weight 2, and
odd non-overlined parts are counted with weight 1.

The proof here is intrinsic.
Let $e_N(t,q) = e_N(t)$ be the generating function for the partitions enumerated by $\beta(m,n)$ with the restriction that the largest part is $\leq N$ (overline the e if $\overline{N}$ is also included).

Then

$$\overline{e}_{4n+3}(t) = tq^{4n+3}\overline{e}_{4n-1} + (1 + tq + t^2q^2 + t^3q^3)e_{4n-1}(tq^2).$$

Now let $n \to \infty$.

$$\begin{aligned}\overline{e}_{\infty}(t) &= (1 + tq + t^2q^2 + t^3q^3)e_{\infty}(tq^2) \\ &= \prod_{n=0}^{\infty}(1 + tq^{2n+1} + t^2q^{4n+2} + t^3q^{6n+3}) \\ &= \sum_{m,n \geq 0} \alpha(m,n)t^m q^n.\end{aligned}$$

There are obvious open questions.

1. Is there an extrinsic proof of the $k = 5$ theorem?
2. Is there an extrinsic proof for $k = 3$ or $k = 4$ in the case of odd parts ocurring $\leq k - 1$ times?
3. What is going on for $k \geq 5$?

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THANK YOU!