# The one-sided cycle shuffles in the symmetric group algebra (abstract)

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full paper: `https://www.cip.ifi.lmu.de/~grinberg/algebra/s2b1.pdf`

We study a new and curious family of elements in the group ring of a symmetric group – or, equivalently, a class of ways to shuffle a deck of cards. Given a probability distribution $P$ on the set $[n] := \{1, 2, \ldots, n\}$, the *one-sided cycle shuffle* corresponding to $P$ consists of picking the card at position $i$ with probability $P(i)$, removing it, and reinserting it at a position weakly below position $i$, chosen uniformly at random. By varying the probability distribution, we obtain an infinite family of shuffling operators.

Formally, we fix a positive integer $n$, and consider the symmetric group $S_n$ on the set $[n]$. For any $k$ distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\text{cyc}_{i_1,i_2,\ldots,i_k}$ denote the permutation in $S_n$ that cycles through $i_1, i_2, \ldots, i_k$ and fixes everything else. (Thus, $\text{cyc}_{i_1,i_2,\ldots,i_k} = \text{id}$ if $k = 1$.)

For each $\ell \in [n]$, we define an element

$$t_\ell := \text{cyc}_\ell + \text{cyc}_{\ell,\ell+1} + \text{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \text{cyc}_{\ell,\ell+1,\ldots,n}$$

of the group ring $\mathbb{R}[S_n]$. We refer to these $n$ elements $t_1, t_2, \ldots, t_n$ as the *somewhere-to-below shuffles*, since the standard interpretation of elements of $\mathbb{R}[S_n]$ in terms of card shuffling allows us to view them as shuffling operators. Namely, applying a given $t_\ell$ to a deck of cards amounts to picking the $\ell$-th card from the top and moving it to a randomly chosen position further down the deck. Note that $t_1$ is the well-known *top-to-random shuffle* studied by Diaconis, Fill, Pitman and others, whereas $t_n = \text{id}$.

While each $t_\ell$ can be viewed as a top-to-random shuffle on a smaller symmetric group, the mutual interaction of $t_1, t_2, \ldots, t_n$ has not been studied to date. Similar families of elements of $\mathbb{R}[S_n]$ include the Young-Jucys-Murphy elements (the transposition shuffles), the Reiner-Saliola-Welker elements (a generalization of the random-to-random shuffle in which several cards are moved) and the Diaconis-Fill-Pitman elements (the top-$m$-to-random shuffles; note that the first of these elements is $t_1$). Unlike the three families just mentioned, the somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$ do not commute. However, they come close to commuting: There is a basis of $\mathbb{R}[S_n]$ on which they all act as upper-triangular matrices; thus, they generate an algebra whose semisimple quotient is commutative (which entails, in particular, that their commutators are nilpotent).

We can, in fact, make this more precise and combinatorial. We define the *descent set* $\text{Des}\, w$ of a permutation $w \in S_n$ to be the set of all $i \in [n-1]$ satisfying $w(i) > w(i+1)$. We let $s_1, s_2, \ldots, s_{n-1}$ be the simple transpositions in $S_n$ (that is, $s_i := \text{cyc}_{i,i+1}$ for each $i \in [n-1]$). For each $I \subseteq [n-1]$, we let $G(I)$ be the subgroup of

$S_n$ generated by the $s_i$ with $i \in I$. Now, for each $w \in S_n$, we set

$$a_w := \sum_{\sigma \in G(\operatorname{Des} w)} w\sigma \in \mathbb{R}[S_n].$$

Then, $(a_w)_{w \in S_n}$ is a basis of the $\mathbb{R}$-vector space $\mathbb{R}[S_n]$, which we call the *descent-destroying basis*. We show that right multiplication by each $t_\ell$ acts as an upper-triangular matrix with respect to this basis – i.e., that each $w \in S_n$ and each $\ell \in [n]$ satisfy

$$a_w t_\ell = \mu_{w,\ell} a_w + (\text{a linear combination of the } a_v \text{ with } v \prec w),$$

where $\mu_{w,\ell}$ is a certain nonnegative integer (which we determine explicitly) and where $\prec$ is a certain partial order on $S_n$. This entails that not only the elements $t_\ell$ but also all their $\mathbb{R}$-linear combinations $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ can be triangularized. In particular, their eigenvalues are rational if the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are.

Better yet, the partial order $\prec$ is an ordinal sum of $f_{n+1}$ antichains, where $(f_0, f_1, f_2, \ldots)$ is the Fibonacci sequence. Thus, right multiplication by an $\mathbb{R}$-linear combination $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ in the basis $(a_w)_{w \in S_n}$ is not just any triangular $n! \times n!$-matrix, but has a convenient block-like structure that causes it to behave better. In particular, the maximum number of distinct eigenvalues of this matrix (obtained when $\lambda_1, \lambda_2, \ldots, \lambda_n$ are generic) is the Fibonacci number $f_{n+1}$. If all these $f_{n+1}$ eigenvalues are distinct, then the matrix is diagonalizable. Both the eigenvalues and their multiplicities are computed explicitly in combinatorial terms.

While we have been working over $\mathbb{R}$ for illustrative purposes, all our proofs hold over any commutative ring (or, for the diagonalizability claim, over any field).

The eigenvalues of shuffling operators have previously gathered attention because of their use in bounding the mixing time of the shuffling schemes (which is the time a shuffling procedure has to be repeated before one gets a thoroughly mixed deck). This works when the transition matrix of the shuffle is symmetric. However, the one-sided cycle shuffles do not satisfy this condition. We overcome this obstacle by giving a strong stationary time, similar to the well-known stopping time for the top-to-random shuffle. For the random-to-below shuffle, in which every card is equally likely to be sent below (i.e., the card shuffling method given by the linear combination $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ with $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/n$), we bound the expected waiting time for this by $n(\log n + \log(\log n) + 1)$.

Several open questions remain. In particular, while the above triangularity result implies that the commutators $[t_i, t_j] = t_i t_j - t_j t_i$ are nilpotent, our numerical experiments suggest that they might be "far more nilpotent" than expected: It appears that any $1 \leq i < j \leq n$ satisfy $[t_i, t_j]^{j-i+1} = 0$ and $[t_i, t_j]^{n-j+1} = 0$. The actions of the $t_\ell$ on Specht modules (irreducible representations of $S_n$) also remain to be explored, as does the structure of the subalgebra generated by $t_1, t_2, \ldots, t_n$.