NEW RESULTS ON RESTRICTION PROBLEM

The restriction problem. Let W_λ denote the irreducible polynomial representation of $GL_n(\mathbf{C})$ indexed by a partition λ with at most n parts, and let V_μ denote the irreducible representation of S_n indexed by a partition μ of n. Restricting W_λ to the subgroup of permutation matrices, which is isomorphic to S_n, we have:

$$\mathrm{res}^{GL_n}_{S_n} W_\lambda(\mathbf{C}^n) = \bigoplus_{\mu} V_\lambda^{\oplus r_{\lambda\mu}}.$$

The multiplicities $r_{\lambda\mu}$ are called the *restriction coefficients*. Finding a combinatorial interpretation of $r_{\lambda\mu}$ is a long standing open problem, see *OPAC 020*.

Littlewood's formula. The best-known way of computing restriction coefficients is by expanding a plethysm of symmetric functions in the basis of Schur functions, the character of W_λ. In [5] Littlewood proved that

$$r_{\lambda\mu} = \langle s_\mu[1 + h_1 + h_2 + \dots], s_\lambda\rangle,$$

where h_i are the complete homogeneous symmetric functions and the plethysm $s_\mu[1 + h_1 + h_2 + \dots]$ may be briefly described as the substitution of the multiset of monomials occuring in $1 + h_1 + h_2 + \dots$ into the variables of the Schur function s_μ.

In [7] we provide a representation-theoretic proof of this Littlewood's formula by obtaining the *Frobenius reciprocity* theorem in the setting of polynomial representation of $GL_n(\mathbf{C})$ and its subgroup S_n.

Recent attempts. Assaf and Speyer [1] and independently, Orellana and Zabrocki [9] introduced *Specht symmetric functions* to study the restriction problem. Orellana, Zabrocki, Saliola and Schilling study the subalgebra of uniform block partitions within the partition algebra in [8], as an intermediate step to considering the restriction problem. Heaton, Sriwongsa and Willenbring prove the positivity of a family of restriction coefficients in [4]. Despite these advances in the problem, a combinatorial formula for restriction coefficients is still unknown.

Our attempts. In [6] we used character polynomials to study the restriction problem. Character polynomials have been used to study characters of families of representations of symmetric groups (see Garsia and Goupil [3]), also used in the context of FI-modules by Church, Ellenberg, and Farb [2]. Note that $r_{\lambda,(n)}$, the multiplicity of the trivial representation of S_n in $W_\lambda(\mathbf{C})$, is the dimension of the space of S_n invariant vectors in $W_\lambda(\mathbf{C})$. Our character polynomial approach answers the following question in a few special cases.
Question: Given partition λ, determine the conditions when $r_{\lambda,(n)} > 0$?

Theorem 0.1. *Let λ be a partition with atmost n parts. We have the following:*

(1) *If λ has two rows then $r_{\lambda,(n)} > 0$ unless $\lambda = (1, 1)$.*
(2) *If λ has two columns then $r_{\lambda,(n)} > 0$ if and only if ${\lambda_1}' - {\lambda_2}' \leq 1$.*
(3) *If $\lambda = (a + 1, 1^b)$ then $r_{\lambda,(n)} > 0$ if and only if $a \geq \binom{b+1}{2}$.*

New results. In ongoing work with Sridhar Narayanan, Amritanshu Prasad and Shraddha Srivastava, we obtain a positive combinatorial rule for the restriction coefficients $r_{\lambda\mu}$ in specific cases. We used *moment generating function* for some GL_n modules, which are proved in [6]. For the proof of the last two theorems, we develop a sign-reversing involution; hence the nature of the proofs is combinatorial. Our results follow.

Theorem 0.2. *Let $\lambda = (k, l)'$, the conjugate of the partition (k, l). Then, for each $n \geq 2$, the sign representation of S_n occurs in $W_{(k,l)'}(\mathbf{C})$ if and only if $(k, l) \in \{(n-1, 0), (n, 0), (n-1, 1), (n, 1)\}$. In all cases it occurs with multiplicity one.*

Theorem 0.3. *For all $a, b \geq 0$, the multiplicity of the sign representation of S_n in $W_{(a+1,1^b)}$ is the number of pairs (λ, μ) such that*

1. *$\lambda = (\lambda_1, \ldots, \lambda_b)$, where $\lambda_1 \geq \cdots \geq \lambda_b \geq 0$,*
2. *$\mu = (\mu_1, \ldots, \mu_{n-b})$, with $\mu_1 > \cdots > \mu_{n-b} \geq 0$,*
3. *$\lambda_1 + \cdots + \lambda_b + \mu_1 + \cdots + \mu_{n-b} = a + 1$,*
4. *$\mu_1 > \lambda_1$.*

Equivalently, the multiplicity is

$$\sum_{\rho \in P(a,n)} \binom{r_\rho}{n-b-1},$$

where $P(a, n)$ denotes the set of partitions of $a + n$ with n non-negative parts, and for a partition $\rho \in P(a, n)$, r_ρ is the number of removable cells of ρ that are not in its first row.

Theorem 0.4. *For all $a, b \geq 0$, the multiplicity of the trivial representation of S_n in $W_{(a+1,1^b)}$ is the number of pairs (λ, μ) of partitions such that*

1. *$\lambda = (\lambda_1, \ldots, \lambda_{n-b})$, where $\lambda_1 \geq \cdots \geq \lambda_{n-b} \geq 0$,*
2. *$\mu = (\mu_1, \ldots, \mu_b)$, with $\mu_1 > \cdots > \mu_b \geq 0$,*
3. *$\lambda_1 + \cdots + \lambda_{n-b} + \mu_1 + \cdots + \mu_b = a + 1$,*
4. *$\mu_1 < \lambda_1 - 1$.*

Equivalently, the multiplicity is

$$\sum_{\rho \in P(a,n)} \binom{r_\rho}{b-1} + \sum_{\rho \in \tilde{P}(a,n)} \binom{r_\rho - 1}{b-1},$$

where $P(a, n)$ denotes the set of partitions of $a + n$ with n non-negative parts, $\tilde{P}(a, n)$ denotes the subset of $P(a, n)$ of partitions whose second-largest part is one less than the largest part, and for a partition $\rho \in P(a, n)$, r_ρ is the number of removable cells of ρ that are not in its first row.

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